Some ideas on bubbly markets

Martin Schweizer

Department of Mathematics, ETH Zürich,
and Swiss Finance Institute

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based on joint work with Martin Herdegen
Motivation
So far, in *incomplete* financial markets, no convincing definition of financial bubbles.

Strong connections in mathematical finance to *strict local* martingales.
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Strong connections in mathematical finance to **strict local martingales**.

Can we make this precise, without putting it into the definition?

How can we avoid the dependence on an a priori choice of numéraire and/or risk-neutral measure?

How can one adjust or define valuation principles to account for the presence of bubbles? [→ work in progress]
Main goals

- Introduce a framework to define what it means that a financial market contains a bubble.
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- Introduce a framework to define what it means that a financial market contains a bubble.
- Deduce from the economic/financial definition a connection to strict local martingales.
- Construct an economically sound valuation principle for financial markets with bubbles. \(\rightarrow\) work in progress
- Keep everything general, and avoid unnecessary assumptions on the financial market.
- Clarify connections to existing ideas and approaches.
Model
Basic setup

- Starting point is an $\mathbb{R}_+^N$-valued adapted RCLL process $\tilde{S} = (\tilde{S}_t)_{t \in [0, T]}$ for a fixed time horizon $T \in (0, \infty)$.

- Vector process $\tilde{S} = (\tilde{S}^1, \ldots, \tilde{S}^N)$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ describes the prices of $N$ basic assets in a financial market.

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Assets are denominated in some abstract non-specified currency unit — and we want a ("numérale-independent") formulation which does not depend on the choice of that unit.

In particular, there is no extra bank account with discounted price 1 — this would be a loss of generality!
We only assume that the sum of all assets never hits 0, and that $S := \tilde{D}\tilde{S}$ is a semimartingale for some strictly positive $\tilde{D}$. 
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Market $S$ is given by all semimartingales of the form $S' = DS$ with $D > 0$ a semimartingale. Each element of $S$ describes asset prices in one possible currency unit.

We want all concepts and results "numéraire-independent" in the sense that they hold for all elements of $S$. 
Self-financing strategies $\vartheta$ are $N$-dimensional predictable $S$-integrable processes and must satisfy (at stopping times $\sigma$)

$$V(\vartheta)(S) := \vartheta^T S = \vartheta_0^T S_0 + \int \vartheta \, dS.$$
Trading strategies — general and good

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- **Allowed** strategies must be self-financing and **undefaultable**, meaning \( V(\vartheta)(S) \geq 0 \). (This is "numéraire-independent".)
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  (Small point to check: integrability of $\vartheta$ for $S$ plus self-financing property implies integrability of $\vartheta$ for $S' = DS$.)
A strategy $\vartheta$ is **maximal** in a class $\Gamma$ if it cannot be improved:

- $\bar{\vartheta} \in \Gamma$ on $[\sigma, T]$ with $V_T(\bar{\vartheta})(S) \geq V_T(\vartheta)(S)$ implies also $V_\sigma(\bar{\vartheta})(S) \geq V_\sigma(\vartheta)(S)$, for stopping time $\sigma \leq T$. 

17 / 72

Martin Schweizer

Some ideas on bubbly markets 9 / 25
Maximality

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  - More precisely: **strong maximality** at $\sigma$ means that
    \[ \forall f \in L^0(\mathcal{F}_T) \setminus \{0\} \text{ such that } \forall \epsilon > 0 \exists \bar{\vartheta} \in \Gamma \text{ with } V_T(\bar{\vartheta})(S) \geq V_T(\vartheta)(S) + f \text{ and } V_\sigma(\bar{\vartheta})(S) \leq V_\sigma(\vartheta)(S) + \epsilon. \]
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The classic setup

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  • *d* risky assets, **one riskless bank account** — so \( N = d + 1 \).
  • Bank account is always 1, risky assets are in units of bank account given by \( X \) — so \( S = (1, X) \). (Maybe \( \bar{D} = (\bar{S}^0)^{-1} \).)
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  - Self-financing strategies are parametrisable by initial wealth $v_0 = \vartheta_0^T S_0$ and $d$-dimensional predictable process $\psi$ — then bank account holdings $\psi^0$ in $\vartheta = (\psi^0, \psi)$ are determined from $v_0$ and $\psi$ by self-financing condition via

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    \[
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    \]
  - Note that **a-admissible**, i.e. $V(v_0, \psi) = v_0 + \int \psi \, dX \geq -a$, is **numéraire-dependent** concept — lower bound is in **units of bank account** $\equiv 1$. Using different units causes problems!
**Absence of arbitrage conditions:**

- **Static viability:** zero strategy 0 is strongly maximal among all allowed **buy-and-hold** strategies.
- **Dynamic viability:** zero strategy 0 is strongly maximal among all allowed strategies.

Note that all these notions are indeed numéraire-independent.
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Stronger forms:

- **Static efficiency:** every allowed buy-and-hold strategy is strongly maximal among all allowed buy-and-hold strategies.
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Examples I

- Example of market which lies outside classic setup
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• Example of single jump process where every strategy is weakly maximal, but not strongly maximal
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- Example of market which lies outside classic setup
- Example of non-maximal strategy
- Example of single jump process where every strategy is weakly maximal, but not strongly maximal
- Example of event tree which is statically viable, but not dynamically viable
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**Finite discrete time:** equivalence between

- dynamic viability
- NA (for undefaultable strategies)

for each numéraire strategy \( \eta \), discounted prices \( S(\eta) = \frac{S}{V(\eta)(S)} \) admit a true EMM.
Results
Numéraire-independent FTAP: (Herdegen)

The market $S$ is dynamically viable (dynamic trading cannot improve upon inactivity/smart traders cannot beat zero) if and only if
**Numéraire-independent FTAP:** (Herdegen)

- The market $S$ is **dynamically viable** (dynamic trading cannot improve upon inactivity/smart traders cannot beat zero) if and only if
- there exists a **pair** $(\eta, Q)$ such that $\eta$ is a numéraire strategy and $Q$ is an equivalent local martingale measure for $V(\eta)$-discounted prices $S^{(\eta)}$ given by $S^{(\eta)} := \frac{S}{V(\eta)(S)}$
Numéraire-independent FTAP: ($\rightarrow$ Herdegen)

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- there exists some $Q$ equivalent to $P$ such that some representative $S^{(\eta)} = \frac{S}{V(\eta)(S)}$ is a $Q$-local martingale.
Numéraire-independent FTAP: \((\longrightarrow \text{Herdegen})\)

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- \(S\) satisfies *NINA* (*numéraire-independent no-arbitrage*).
Key aspects:

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Key aspects:

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Key aspects:

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- Market $S$ is described by one (semimartingale) representative $S$ — but this plays no special role.

- *$S$ itself need not admit an ELMM*.

- There is some representative $S' \in S$ which admits an ELMM.
Recall: **viable** means zero cannot be improved, **efficient** that any buy-and-hold cannot be improved. Connection?
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**No dominance:** for each asset $i = 1, \ldots, N$, look at the buy-and-hold strategy $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ of holding one unit of asset $i$. Sum is market portfolio $\eta^S = (1, \ldots, 1)$.

- **Static no dominance:** for each stopping time $\sigma$, $\eta^S$ is weakly maximal on $[\sigma, T]$ among all allowed **buy-and-hold** strategies.
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- **Dynamic no dominance:** for each stopping time $\sigma$, $\eta^S$ is weakly maximal on $[\sigma, T]$ among all allowed strategies.
Efficiency and no dominance

- Equivalent to **dynamic no dominance**: for each asset $i = 1, \ldots, N$, $e_i$ is weakly maximal on $[0, T]$ among all allowed strategies.
  - Precise and minimal formulation for original idea of Merton.
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- **Connection** (for both static and dynamic versions):
  - $S$ is **efficient** if and only if
  - $S$ is **viable plus** satisfies **no dominance**.
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  - **no buy-and-hold** strategy can be improved.
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- Put differently, for dynamic case: If 0 and market portfolio $\eta^S$ cannot be improved by dynamic trading, then (→ Merton)
  - **no buy-and-hold** strategy can be improved.
  - **not even any bounded dynamic** strategy can be improved.
FTAP, no dominance and efficiency:

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  - $S$ satisfies **NINA** (numéraire-independent no-arbitrage)
  - plus dynamic no dominance
  - if and only if

FTAP and numéraires
Efficiency and no dominance
Bubbly markets
Examples II
FTAP, no dominance and efficiency:

- $S$ satisfies **NINA** (numéraire-independent no-arbitrage)
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  if and only if
- there exists a pair $(\eta, Q)$ such that $\eta$ is a numéraire strategy
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FTAP, no dominance and efficiency:

- $S$ satisfies **NINA** (numéraire-independent no-arbitrage) plus dynamic no dominance if and only if there exists a pair $(\eta, Q)$ such that $\eta$ is a numéraire strategy and $Q$ is an equivalent true martingale measure for $V(\eta)$-discounted prices $S^{(\eta)}$ given by $S^{(\eta)} := \frac{S}{V(\eta)(S)}$ if and only if
- $S$ is dynamically efficient.
FTAP, no dominance and efficiency:

- **FTAP, no dominance and efficiency:**
  - \( S \) satisfies **NINA (numéraire-independent no-arbitrage)**
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  - if and only if
  - there exists a **pair** \((\eta, Q)\) such that \( \eta \) is a **numéraire strategy**
    - and \( Q \) is an **equivalent true martingale measure** for
    - \( V(\eta) \)-discounted prices \( S(\eta) \) given by \( S(\eta) := \frac{S}{V(\eta)(S)} \)
  - if and only if
  - \( S \) is dynamically efficient.

(Proofs rest on dual characterisations via martingale properties.)
Bubbly market: $S$ is **not dynamically efficient** (*dynamic trading can be used to improve upon buy-and-hold*).
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- Smart traders can do better than static traders.
Bubbly market: $S$ is not dynamically efficient (dynamic trading can be used to improve upon buy-and-hold).

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Nontrivial bubbly market: in addition dynamically viable (dynamic trading will not improve inactivity) and statically efficient (buy-and-hold cannot be improved by buy-and-hold).
Bubbly market: $S$ is not dynamically efficient (*dynamic trading can be used to improve upon buy-and-hold*).

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Nontrivial bubbly market: in addition dynamically viable (*dynamic trading will not improve inactivity*) and statically efficient (*buy-and-hold cannot be improved by buy-and-hold*).

- Smart traders cannot beat inactivity.
- Static traders cannot beat other static traders.
- But — smart traders can do better than static traders.
Bubbly markets:

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Bubbly markets:

- $S$ is a **bubbly market** *(not dynamically efficient)*
  
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(Based on rewriting definitions and using their connection.)
Bubbly markets:

- **Bubbly markets:**
  - $S$ is a **bubbly market** \((not\ dynamically\ efficient)\)
    if and only if
  - either $S$ admits **some numéraire-independent arbitrage** \((is\ not\ dynamically\ viable)\)
  - or for **some** asset $i \in \{1, \ldots, N\}$, the **buy-and-hold strategy** $e_i$ in asset $i$ is **not maximal** among all allowed strategies \((S\ fails\ to\ satisfy\ dynamic\ no\ dominance)\).
Bubbly markets:

- **S** is a **bubbly market** (*not dynamically efficient*) if and only if
  - either **S** admits **some numéraire-independent arbitrage** (*is not dynamically viable*)
  - or for **some** asset $i \in \{1, \ldots, N\}$, the **buy-and-hold strategy** $e_i$ in asset $i$ **is not maximal** among all allowed strategies (**S** fails to satisfy **dynamic no dominance**).

(This is just rewriting the definitions and using their connection.)
Suppose that \( S \) is a **nontrivial bubbly market**. Then:
Suppose that $S$ is a nontrivial bubbly market. Then:

- There exists a pair $(\eta, Q)$ such that $\eta$ is a numéraire strategy and $V(\eta)$-discounted prices $S(\eta)$ given by $S(\eta) := \frac{S}{V(\eta)(S)}$ are a strict local martingale under $Q$. 
Suppose that $S$ is a **nontrivial bubbly market**. Then:

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  $$S^{(\eta)} := \frac{S}{\epsilon(\eta)(S)}$$
  are a **strict local martingale** under $Q$.

- For **every** $Q$ equivalent to $P$, and for **every** numéraire strategy $\eta$ such that $V(\eta)$-discounted prices $S^{(\eta)}$ are a local martingale under $Q$, $S^{(\eta)}$ is a **strict local martingale** under $Q$. 
Suppose that $S$ is a **nontrivial bubbly market**. Then:

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- For every $Q$ equivalent to $P$, and for every numéraire strategy $\eta$ such that $V(\eta)$-discounted prices $S^{(\eta)}$ are a local martingale under $Q$, $S^{(\eta)}$ is a **strict local martingale** under $Q$.

- In that sense, a nontrivial bubbly market is a **robust model of a bubble market**.
Simple example of a nontrivial bubbly market:

- $(1, S^{(2)})$ where $Y := 1/S^{(2)}$ is a local $P$-martingale
- $Y$ has predictable representation property
- $Y$ is strict local $P$-martingale
Examples II

- Simple example of a nontrivial bubbly market:
  - \((1, S^{(2)})\) where \(Y := 1/S^{(2)}\) is a local \(P\)-martingale
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- Concrete example of a nontrivial bubbly market which is incomplete (CEV model with suitable stochastic volatility)
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- Concrete example of a market where \(S\) is a strict local martingale under some \(Q\), and a true martingale under another \(Q'\) (non-robust bubble market)
... still remain to be worked out in detail ...
The end

Thank you for your attention

http://www.math.ethz.ch/~mschweiz

or google “Martin Schweizer”