Some ideas on bubbly markets

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Motivation

Basic questions

- So far, in **incomplete** financial markets, no convincing definition of financial bubbles.
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- Strong connections in mathematical finance to **strict local martingales**.
- Can we make this precise, without putting it into the *definition*?
- How can we avoid the dependence on an a priori choice of numéraire and/or risk-neutral measure?
- How can one adjust or define valuation principles to account for the presence of bubbles? *[$\longrightarrow$ work in progress]*

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- **Deduce** from the economic/financial definition a connection to **strict local martingales**.
- Construct an economically sound valuation principle for financial markets with bubbles. *[⟶ work in progress]*
- Keep everything general, and avoid unnecessary assumptions on the financial market.
- Clarify connections to existing ideas and approaches.

Model

Basic setup

- Starting point is an $\mathbb{R}_+^N$-valued adapted RCLL process $\tilde{S} = (\tilde{S}_t)_{t \in [0,T]}$ for a fixed time horizon $T \in (0, \infty)$.
- Vector process $\tilde{S} = (\tilde{S}^1, \ldots, \tilde{S}^N)$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ describes the prices of N **basic assets** in a financial market.
- Assets are denominated in some abstract non-specified currency unit — and we want a *("numéraire-independent")* formulation which does not depend on the choice of that unit.

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- Assets are denominated in some abstract non-specified currency unit — and we want a *("numéraire-independent")* formulation which does not depend on the choice of that unit.
- In particular, there is **no extra bank account** with discounted price 1 — this would be a loss of generality!

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- Market $\mathcal{S}$ is given by all semimartingales of the form $S' = DS$ with $D > 0$ a semimartingale. Each element of $\mathcal{S}$ describes asset prices in one possible currency unit.
- We want all concepts and results *"numéraire-independent"* in the sense that they hold for all elements of $\mathcal{S}$.

Trading strategies — general and good

- **Self-financing strategies** ϑ are N-dimensional predictable S-integrable processes and must satisfy (at stopping times σ)

$$V(\vartheta)(S) := \vartheta^\top S = \vartheta_0^\top S_0 + \int \vartheta \, dS.$$

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- **Allowed** strategies must be self-financing and **undefaultable**, meaning $V(\vartheta)(S) \geq 0$. (This is *"numéraire-independent"*.)

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  - (Small point to check: integrability of ϑ for S plus self-financing property implies integrability of ϑ for $S' = DS$.)

Maximality

- A strategy ϑ is **maximal** in a class Γ if it cannot be improved:
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  - More precisely: **strong maximality** at σ means that $\nexists f \in L^0(\mathcal{F}_T) \setminus \{0\}$ such that $\forall \epsilon > 0 \; \exists \bar{\vartheta} \in \Gamma$ with $V_T(\bar{\vartheta})(S) \geq V_T(\vartheta)(S) + f$ and $V_\sigma(\bar{\vartheta})(S) \leq V_\sigma(\vartheta)(S) + \epsilon$.

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  - (For completeness: **weak maximality** at σ means that $\nexists f \in L^0(\mathcal{F}_T) \setminus \{0\}$ such that $\exists \bar{\vartheta} \in \Gamma$ with $\forall \epsilon > 0$, $V_T(\bar{\vartheta})(S) \geq V_T(\vartheta)(S) + f$ and $V_\sigma(\bar{\vartheta})(S) \leq V_\sigma(\vartheta)(S) + \epsilon$.

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  - Note that **a-admissible**, i.e. $V(v_0, \psi) = v_0 + \int \psi \, dX \geq -a$, is **numéraire-dependent** concept — lower bound is **in units of bank account** $\equiv 1$. Using different units causes problems!

Concepts

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- Note that all these notions are indeed *numéraire-independent*.

Examples I

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- Example of event tree which is statically viable, but not dynamically viable

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- **Finite discrete time:** equivalence between
  - dynamic viability
  - NA (for undefaultable strategies)
  - for each numéraire strategy η, discounted prices $S^{(\eta)} = \frac{S}{V(\eta)(S)}$ admit a true EMM.

Results

FTAP and numéraires

- **Numéraire-independent FTAP:** ($\longrightarrow$ *Herdegen*)
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- $\mathcal{S}$ satisfies **NINA (numéraire-independent no-arbitrage)**.

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- *S itself need not admit an ELMM.*
- There is some representative $S' \in \mathcal{S}$ which admits an ELMM.

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  - **Static no dominance:** for each stopping time σ, $\eta^{\mathcal{S}}$ is weakly maximal on $[\![\sigma, T]\!]$ among all allowed **buy-and-hold** strategies.

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  - **no buy-and-hold** strategy can be improved.
  - **not even any bounded dynamic** strategy can be improved.

FTAP, no dominance and efficiency

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- (Proofs rest on dual characterisations via martingale properties.)

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- **Nontrivial bubbly market:** in addition **dynamically viable** *(dynamic trading will not improve inactivity)* and **statically efficient** *(buy-and-hold cannot be improved by buy-and-hold)*.
  - Smart traders cannot beat inactivity.
  - Static traders cannot beat other static traders.
  - But — smart traders can do better than static traders.

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- (This is just rewriting the definitions and using their connection.)

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  - For **every** Q equivalent to P, and for **every** numéraire strategy η such that $V(\eta)$-discounted prices $S^{(\eta)}$ are a local martingale under Q, $S^{(\eta)}$ is a **strict local martingale** under Q.

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- In that sense, a nontrivial bubbly market is **a robust model of a bubble market**.

Examples II

- Simple example of a nontrivial bubbly market:
  - $(1, S^{(2)})$ where $Y := 1/S^{(2)}$ is a local P-martingale
  - Y has predictable representation property
  - Y is strict local P-martingale

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- Concrete example of a nontrivial bubbly market which is incomplete (CEV model with suitable stochastic volatility)
- Concrete example of a market where S is a **strict local** martingale under some Q, and a **true** martingale under another Q' *(non-robust bubble market)*

Valuation principles

- . . . still remain to be worked out in detail . . .

The end

Thank you for your attention

http://www.math.ethz.ch/∼mschweiz

or google "Martin Schweizer"