The FTAP and robust superhedging in continuous time for continuous paths

Sara Biagini
University of Pisa
(joint with Bruno Bouchard, Kostas Kardaras and Marcel Nutz)

Banff International Research Station

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Underlying framework

State space E (Polish), denoting possible states in economy.

- Append isolated cemetery state $\triangle$ to E.
- For right-continuous $\omega : \mathbb{R}_+ \mapsto E \cup \{\triangle\}$, define

$$\zeta(\omega) := \inf \{t \in \mathbb{R}_+ \mid \omega_t = \triangle\}$$

  to be the lifetime of ω.

Filtered space: $(\Omega, \mathbf{F})$, where

- $\Omega =$ all right-continuous ωs such that $\omega_0 \in E$, ω continuous on $[0, \zeta(\omega))$ and $\omega = \triangle$ on $[\zeta(\omega), \infty)$.
- $\mathbf{F}$: natural filtration of ω.

Prior-to-zeta equivalence

$\mathbb{Q}$ and $\mathbb{P}$ are **prior-to-ζ equivalent probabilities** if

$$\mathbb{Q}_{|\mathcal{F}_t \cap \{\zeta > t\}} \sim \mathbb{P}_{|\mathcal{F}_t \cap \{\zeta > t\}} \text{ for all } t \in \mathbb{R}_+.$$

Notation: $\mathbb{Q} \overset{<\zeta}{\sim} \mathbb{P}$

Financial market

Class of models $\mathcal{P}$. We require that $\zeta = \infty$, a.s. for all $\mathbb{P} \in \mathcal{P}$.

Asset prices S d-dimensional, adapted, continuous paths.

Simple trading from capital $x \in (0, \infty)$:

$$\mathcal{H}_s(x) := \{H \mid H \text{ simple predict.}, \; x + H \bullet S \geq 0, \; a.s. \text{ for all } \mathbb{P} \in \mathcal{P}\}.$$

Superhedging price from simple trading: for $T \in \mathbb{R}_+$ and $G \in \mathcal{L}_+(\mathcal{F}_T)$,

$$v_s(T, G) := \inf\{x > 0 \mid \exists H \in \mathcal{H}_s(x) \text{ with } x + (H \bullet S)_T \geq G, \; a.s. \, \forall \mathbb{P}\}$$

No arbitrage and prior-to-zeta ELMM

No Arbitrage of the 1st kind (NA$_1$)

For all $T \in \mathbb{R}_+$ and $G \in \mathcal{L}_+(\mathcal{F}_T)$,

$$v_s(T, G) = 0 \quad \Longrightarrow \quad G = 0, \ a.s. \text{ for all } \mathbb{P} \in \mathcal{P}$$

Prior-to-ζ ELMM $\mathbb{Q}$ corresponding to a fixed $\mathbb{P} \in \mathcal{P}$:

1. $\mathbb{Q} \overset{<\zeta}{\sim} \mathbb{P}$.
2. $\exists$ sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that:
   - $\mathbb{Q}[\tau_n < \zeta] = 1$ for all $n \in \mathbb{N}$, and $\mathbb{Q}[\lim_{n \to \infty} \tau_n = \zeta] = 1$.
   - $(S_{\tau_n \wedge t})_{t \in \mathbb{R}_+}$ is a $\mathbb{Q}$-mart for all $n \in \mathbb{N}$.

The nondominated FTAP

Notation: $\mathcal{Q}^{\mathbb{P}}$ class of prior-to-zeta ELMM with fixed $\mathbb{P} \in \mathcal{P}$.

The (version of the) FTAP is

$$\mathrm{NA}_1 \iff \forall \mathbb{P} \in \mathcal{P}, \; \mathcal{Q}^{\mathbb{P}} \neq \emptyset.$$

Corollary. NA_1 implies S is semimartingale for all $\mathbb{P} \in \mathcal{P}$.

Robust superhedging duality

(Extended) Trading from capital $x \in (0, \infty)$:

$$\mathcal{H}(x) := \left\{ H \in L(S; \mathcal{P}, \mathbf{F}^{\mathcal{P}}) \mid x + H \bullet S \geq 0, \ \mathbb{P}\text{-a.s.}, \ \forall \mathbb{P} \in \mathcal{P} \right\}.$$

Superhedging value: for $T \in \mathbb{R}_+$ and $G \in \mathcal{L}_+(\mathcal{F}_T)$,

$$v(T, G) = \inf \{ x \mid \exists H \in \mathcal{H}(x) \text{ with } x + (H \bullet S)_T \geq G, \ \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P} \}.$$

Robust superhedging duality: Assume NA_1, and structure on $\mathcal{P}$.
For all $T \in \mathbb{R}_+$ and $G \in \mathcal{L}_+(\mathcal{F}_T)$,

$$v(T, G) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \left[G \mathbf{1}_{\{T < \zeta\}} \right].$$

where $\mathcal{Q} = \bigcup_{\mathbb{P} \in \mathcal{P}} \mathcal{Q}^{\mathbb{P}}$. Also,

$$v(T, G) < \infty \Rightarrow \exists \text{ optimal } H^* \in \mathcal{H}(v(T, G))$$

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