The Limits of Leverage

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The Limits of Leverage

Efficient Frontier

- Average Return ($y$) against volatility ($x$) as benchmarks’ multiples.
- No transaction costs at zero (0,0) or full investment (1,1).
- Higher Leverage = Lower Sharpe Ratio.
- Maximum return at around 9x leverage. More leverage decreases return.

$\mu = 8\%, \sigma = 16\%, \varepsilon = 1\%$
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Unlimited Leverage?

“If an investor can borrow or lend as desired, any portfolio can be levered up or down. A combination with a proportion $k$ invested in a risky portfolio and $1 - k$ in the riskless asset will have an expected excess return of $k$ and a standard deviation equal to $k$ times the standard deviation of the risky portfolio.” — Sharpe (2011)

- **Implications:**
  - Efficient frontier linear. One Sharpe ratio.
  - Any efficient portfolio spans all the others.
  - Portfolio choice meaningless for risk-neutral investors.

- **Applications:**
  - Levered and inverse ETFs: up to 3x and -3x leverage. A 10x ETF?
  - Leverage to increase returns from small mispricings.
  - Capital ratios as regulatory leverage constraints.

- **Limitations:**
  - Constant leverage needs constant trading. Rebalancing costs?
  - Higher beta with lower alpha (Frazzini and Pedersen, 2013).
  - Levered ETFs on illiquid indexes have substantial tracking error.
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What We Do

● Model
  ● Maximize long-term return given average volatility.
  ● Constant proportional bid-ask spread.
  ● IID returns. Geometric Brownian motion.
  ● Continuous trading allowed. No constraints.

● Results
  ● Sharpe ratio declines as leverage increases.
  ● Limits of leverage.
  ▪ Beyond a certain threshold, even expected return declines.
  ● Leverage Multiplier.
  ▪ Maximum factor by which the asset return can be increased:
    \[
    0.3815 \left( \frac{\mu}{\sigma^2} \right)^{1/2} \varepsilon^{-1/2}
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    \[\varepsilon\] bid-ask spread, \[\mu\] excess return, \[\sigma\] volatility.
  ● Optimal tradeoff between alpha and tracking error.
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      where \(\varepsilon\) is bid-ask spread, \(\mu\) excess return, and \(\sigma\) volatility.
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More Volatility = Cheaper Leverage

- Average Return (y) against volatility (x), annualized.
- Frontier of asset with 10% return with 20% volatility superior to that of asset with 5% return and 10% volatility.
- More asset volatility = less rebalancing costs for same portfolio volatility.

\[ \mu / \sigma = 0.5, \varepsilon = 1\%, \sigma = 10\%(bottom), 20\%, 50\%(top) \]
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Sharpe ratio $\mu/\sigma = 0.5$

- Approximate value $\approx 0.3815 \left( \frac{\mu}{\sigma^2} \right)^{1/2} \varepsilon^{-1/2}$
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\[ \text{Volatility (} \sigma \text{)} \quad 0.01\% \quad 0.10\% \quad 1.00\% \]

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Market

- Safe rate $r$. Geometric Brownian motion for ask price $S_t$. Bid Price = 
  \[(1 - \varepsilon)S_t\]
  \[
  \frac{dS_t}{S_t} = (\mu + r)dt + \sigma dB_t, \quad S_0, \sigma, \mu > 0,
  \]

- $\varphi_t = \varphi_t^\uparrow - \varphi_t^\downarrow$ number of shares at time $t$ as purchases minus sales.

- Fund value at ask prices:
  \[
  dw_t = rw_t dt + \varphi_t dS_t - \varepsilon d\varphi_t^\downarrow
  \]

- Solvency constraint $w_t - \varepsilon(\varphi_t)^+ S_t \geq 0$ a.s. for all $t \geq 0$. 
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Return and Volatility

• Usual fund performance statistics in terms of returns \( r_t = \frac{w_t - w_{t-\Delta t}}{w_{t-\Delta t}} \)

• Average return on \([0, T]\)

\[
\bar{r}_T = \frac{1}{T} \sum_{t=k\Delta t}^{0 \leq t \leq T} r_t \approx \frac{1}{T} \int_0^T \frac{dw_t}{w_t}
\]

Without trading costs, \( \frac{1}{T} \int_0^T \frac{dw_t}{w_t} = \frac{1}{T} \int_0^T \mu\pi_t dt + \frac{1}{T} \int_0^T \sigma\pi_t dW_t \), with \( \pi_t \) portfolio weight.

• Average volatility on \([0, T]\)

\[
\frac{1}{T} \sum_{t=k\Delta t}^{0 \leq t \leq T} (r_t - \bar{r}_T \Delta t)^2 \approx \frac{1}{T} \int_0^T \frac{d\langle w\rangle_t}{w_t^2}
\]

Without trading costs, \( \frac{1}{T} \int_0^T \frac{d\langle w\rangle_t}{w_t^2} = \frac{\sigma^2}{T} \int_0^T \pi_t^2 dt \).
Return and Volatility

- Usual fund performance statistics in terms of returns \( r_t = \frac{w_t - w_{t-\Delta t}}{w_{t-\Delta t}} \)
- Average return on \([0, T]\)

\[
\bar{r}_T = \frac{1}{T} \sum_{t=k\Delta t}^{0 \leq t \leq T} r_t \approx \frac{1}{T} \int_0^T \frac{d w_t}{w_t}
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Return and Volatility

• Usual fund performance statistics in terms of returns $r_t = \frac{w_t - w_{t-\Delta t}}{w_t - \Delta t}$

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$$\bar{r}_T = \frac{1}{T} \sum_{t=k\Delta t}^{0\leq t \leq T} r_t \approx \frac{1}{T} \int_0^T dw_t \frac{w_t}{w_t}$$

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Without trading costs, $\frac{1}{T} \int_0^T \frac{d\langle w \rangle_t}{w_t^2} = \frac{\sigma^2}{T} \int_0^T \pi_t^2 dt$. 
Objective

- Maximize return-volatility tradeoff for large $T$

$$E \left[ \frac{1}{T} \left( \int_0^T \frac{dw_t}{w_t} - \frac{\gamma}{2} \int_0^T \frac{d\langle w \rangle_t}{w_t^2} \right) \right]$$

- Equals to

$$r + \frac{1}{T} E \left[ \int_0^T \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t}{\varphi_t} \right]$$

- Without trading costs, usual mean-variance portfolio $\pi_t = \frac{\mu}{\gamma \sigma^2}$ optimal. Infinite rebalancing costs.
- $\gamma = 0$: maximize return, forget volatility. Ill-posed without trading costs.
- $\gamma = 1$: logarithmic utility. Taksar et al. (1988), Gerhold et al. (2012).

- Tradeoff between high leverage and high trading costs. Well-posed even without risk.
Objective

• Maximize return-volatility tradeoff for large $T$

$$E \left[ \frac{1}{T} \left( \int_0^T \frac{d\pi_t}{\pi_t} - \frac{\gamma}{2} \int_0^T \frac{d\langle w \rangle_t}{w_t^2} \right) \right]$$

• Equals to

$$r + \frac{1}{T} E \left[ \int_0^T \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^{\uparrow}}{\varphi_t} \right]$$

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- Maximize return-volatility tradeoff for large $T$

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The Limits of Leverage

Efficient Frontier ($\gamma > 0$)

**Theorem**

Trade to keep portfolio weight $\pi_t$ within boundaries $\pi_-$ (buy) and $\pi_+$ (sell)

$$\pi_\pm = \frac{\zeta_\pm}{1 + \zeta_\pm} = \pi_* \pm \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2\right)^{1/3} \varepsilon^{1/3} - (\gamma - 1) \left(\frac{\pi_* (1 - \pi_*)}{6\gamma^2}\right)^{2/3} \varepsilon^{2/3} + O(\varepsilon)$$

where $\pi_* = \mu/(\gamma \sigma^2)$ and $\zeta_\pm$ solve the free-boundary problem $(W, \zeta_-, \zeta_+)$

$$\frac{1}{2} \sigma^2 \zeta^2 W''(\zeta) + (\sigma^2 + \mu) \zeta W'(\zeta) + \mu W(\zeta) - \frac{1}{(1+\zeta)^2} \left(\mu - \frac{\gamma \sigma^2 \zeta}{1+\zeta}\right) = 0,$$

$W(\zeta_-) = 0, \quad W(\zeta_+) = \frac{\varepsilon}{(1+\zeta_+)(1+(1-\varepsilon)\zeta_+)}$,

$W'(\zeta_-) = 0, \quad W'(\zeta_+) = \frac{\varepsilon((1-\varepsilon)\zeta_+^2 - 1)}{(1+\zeta_+)^2(1+(1-\varepsilon)\zeta_+)^2}$

- Solution similar to utility maximization. Same first-order approximation.
- No-trade region around the frictionless portfolio.
- Result valid for $\varepsilon$ small enough.
Efficient Frontier \((\gamma > 0)\)

**Theorem**

*Trade to keep portfolio weight \(\pi_t\) within boundaries \(\pi_-\) (buy) and \(\pi_+\) (sell)*

\[
\pi_\pm = \frac{\zeta_\pm}{1 + \zeta_\pm} = \pi_* \pm \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2\right)^{1/3} \varepsilon^{1/3} - (\gamma - 1) \left(\frac{\pi_* (1 - \pi_*)}{6\gamma^2}\right)^{2/3} \varepsilon^{2/3} + O(\varepsilon)
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\]

\[
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\[
\begin{align*}
\frac{1}{2} \sigma^2 \zeta^2 W''(\zeta) + (\sigma^2 + \mu) \zeta W'(\zeta) + \mu W(\zeta) - \frac{1}{(1+\zeta)^2} \left( \mu - \frac{\gamma \sigma^2 \zeta}{1+\zeta} \right) &= 0, \\
W(\zeta_-) &= 0, \quad W(\zeta_+) = \frac{\varepsilon}{(1+\zeta_+)(1+(1-\varepsilon)\zeta_+)} , \\
W'(\zeta_-) &= 0, \quad W'(\zeta_+) = \frac{\varepsilon((1-\varepsilon)\zeta_+^2 - 1)}{(1+\zeta_+)^2(1+(1-\varepsilon)\zeta_+)^2}
\end{align*}
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Limits of Leverage ($\gamma = 0$)

Theorem

*Trade to keep portfolio weight $\pi_t$ within boundaries $\pi_-$ (buy) and $\pi_+$ (sell)*

$$
\pi_\pm = \frac{\zeta_\pm}{1 + \zeta_\pm} = B_\pm \kappa^{1/2} (\mu/\sigma^2)^{1/2} \varepsilon^{-1/2} + 1 + O(\varepsilon^{1/2}),
$$

where $B_- = (1 - \kappa)$, $B_+ = 1$ and $\kappa \approx 0.5828$ is the root of $\frac{3}{2}\kappa + \log(1 - \kappa) = 0$. $\zeta_\pm$ solve the free-boundary problem $(W, \zeta_-, \zeta_+)$

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$$

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- Frictionless problem meaningless. Infinite leverage.
- Pure tradeoff between leverage and rebalancing costs.
- $\pi_-$ is the **multiplier**. Maximum return is $\mu \pi_-$. 
- Approximate relation $\frac{\pi_-}{\pi_+} \approx 0.4172$. 
The Limits of Leverage

Limits of Leverage ($\gamma = 0$)

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Does it make sense?

- As risk-aversion vanishes, do solutions converge to risk-neutral ones?
- Spread of $\varepsilon$ implies maximum leverage of $1/\varepsilon$. Is this driving results?

Assumption

For any $\gamma \in [0, \bar{\gamma}]$ and $\varepsilon = \bar{\varepsilon}$ the free boundary problem has a solution.

Lemma (Convergence)

Under the assumption, the solution for $\gamma = 0$ and $\varepsilon = \bar{\varepsilon}$ coincides with the limit for $\gamma \downarrow 0$ of the solutions for the same $\bar{\varepsilon}$.

Lemma (Interior solution)

Under the assumption, the optimal strategy is interior:

$$\pi_+ < \frac{1}{\varepsilon}.$$
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*For any $\gamma \in [0, \bar{\gamma}]$ and $\varepsilon = \bar{\varepsilon}$ the free boundary problem has a solution.*

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Buy (bottom) and Sell (top) boundaries (y) vs. volatility (x), as multiples.
Trivial at zero (0,0) or full investment (1,1).
Boundaries finite even for $\gamma = 0$ or $\gamma < 0$. 

$\mu = 8\%$, $\sigma = 16\%$, $\varepsilon = 1\%$
• Buy (bottom) and Sell (top) boundaries (y) vs. volatility (x), as multiples.
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$\mu = 8\%, \sigma = 16\%, \varepsilon = 1\%$
Tracking Levered Portfolios

- **Fund return** $r_t^F$, **benchmark return** $r_t^B$, **target exposure** $\pi_*$. $w$ for value.
- **Realized Alpha**

$$\bar{\alpha}_T = \frac{1}{n\Delta t} \sum_{0 \leq t \leq T} (r_t^F - \pi_* r_t^B) \approx \frac{1}{T} \int_0^T \left( \frac{dw_t^F}{w_t^F} - \pi_* \frac{dw_t^B}{w_t^B} \right)$$

- **Realized Tracking Error** $\bar{s}$

$$\bar{s}^2 = \frac{1}{n\Delta t} \sum_{0 \leq t \leq T} (r_t^F - \pi_* r_t^B - \bar{\alpha}_T)^2 \approx \frac{1}{T} \left< \int_0^T \left( \frac{dw_t^F}{w_t^F} - \pi_* \frac{dw_t^B}{w_t^B} \right) \right>_T$$

- **Maximize Alpha with tracking error constraint**

$$\frac{1}{T} E \left[ \int_0^T \left( (\mu + \gamma \sigma^2 \pi_*) \pi_t - \frac{\gamma}{2} \sigma^2 \pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t^\dagger}{\varphi_t} \right] - \mu \pi_* - \frac{\gamma}{2} \sigma^2 \pi_*^2$$

- **Equivalent to previous objective, but with** $\tilde{\mu} = \mu + \gamma \sigma^2 \pi_*$. 
Tracking Levered Portfolios

- Fund return $r_t^F$, benchmark return $r_t^B$, target exposure $\pi_*$. $w$ for value.

- Realized Alpha

$$\bar{\alpha}_T = \frac{1}{n\Delta t} \sum_{0 \leq t \leq T} (r_t^F - \pi_* r_t^B) \approx \frac{1}{T} \int_0^T \left( \frac{dw_t^F}{w_t^F} - \pi_* \frac{dw_t^B}{w_t^B} \right)$$

- Realized Tracking Error $\bar{s}$

$$\bar{s}^2 = \frac{1}{n\Delta t} \sum_{0 \leq t \leq T} (r_t^F - \pi_* r_t^B - \bar{\alpha}_T)^2 \approx \frac{1}{T} \left\langle \int_0^T \left( \frac{dw_t^F}{w_t^F} - \pi_* \frac{dw_t^B}{w_t^B} \right) \right\rangle_T$$

- Maximize Alpha with tracking error constraint

$$\frac{1}{T} E \left[ \int_0^T \left( (\mu + \gamma \sigma^2 \pi_*) \pi_t - \gamma \frac{\sigma^2}{2} \pi_t^2 \right) \, dt - \varepsilon \int_0^T \pi_t \frac{d\varphi^+_t}{\varphi_t} \right] - \mu \pi_* - \frac{\gamma}{2} \sigma^2 \pi_*^2$$

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$$\frac{1}{T} \mathbb{E} \left[ \int_0^T \left( \mu + \gamma \sigma^2 \pi_* \right) \pi_t - \frac{\gamma}{2} \sigma^2 \pi_t^2 \right] dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t}{\varphi_t} - \mu \pi_* - \frac{\gamma}{2} \sigma^2 \pi_*^2$$

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The Limits of Leverage

• Fund return $r_t^F$, benchmark return $r_t^B$, target exposure $\pi_*$. $w$ for value.
Tracking Levered Portfolios

- Fund return $r_t^F$, benchmark return $r_t^B$, target exposure $\pi_*$. $w$ for value.
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$$\bar{\alpha}_T = \frac{1}{n\Delta t} \sum_{0 \leq t \leq T} (r_t^F - \pi_* r_t^B) \approx \frac{1}{T} \int_0^T \left( \frac{dw_t^F}{w_t^F} - \pi_* \frac{dw_t^B}{w_t^B} \right)$$

- Realized Tracking Error $\bar{s}^2$

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$$\frac{1}{T} \mathbb{E} \left[ \int_0^T \left( (\mu + \gamma \sigma^2 \pi_*) \pi_t - \frac{\gamma}{2} \sigma^2 \pi_t^2 \right) dt - \varepsilon \int_0^T \pi_t \frac{d\varphi_t}{\varphi_t} \right] - \mu \pi_* - \frac{\gamma}{2} \sigma^2 \pi_*^2$$

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- Equivalent to previous objective, but with $\tilde{\mu} = \mu + \gamma \sigma^2 \pi_*$. 
Alpha vs. Tracking Error

Theorem

For the $R^2 = \lim_{T \to \infty} \left( \frac{1}{T} \int_0^T \pi_t dt \right)^2$ of a fund with target $\pi^*$ and risk-aversion $\gamma$

$$\sqrt{1 - R^2} = \frac{\sqrt{3} |1 - \pi^*|}{6} \left( \frac{6}{\gamma \pi^* (1 - \pi^*)} \right)^{1/3} \varepsilon^{1/3} + O(\varepsilon)$$

Alpha is excess exposure minus average trading cost

$$\bar{\alpha} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \mu(\pi_t - \pi^*) dt - \varepsilon \pi_t \frac{d\varphi_t}{\varphi_t} \right) = -\frac{3\sigma^2}{\gamma} \left( \frac{\gamma \pi^* (1 - \pi^*)}{6} \right)^{4/3} \varepsilon^{2/3} + O(\varepsilon)$$

whence

$$\bar{\alpha} \approx -\frac{\sqrt{3}}{12} \sigma^2 \pi^* (1 - \pi^*)^2 \frac{\varepsilon}{\sqrt{1 - R^2}}.$$

- Optimal tradeoff between alpha and tracking error.
- With low $\gamma$ lower costs (higher alpha), but more tracking error (lower $R^2$).
- With high $\gamma$ high $R^2$ but also high trading costs.
**Alpha vs. Tracking Error**

**Theorem**

For the $R^2 = \lim_{T \to \infty} \left( \frac{1}{T} \int_0^T \pi_t dt \right)^2$ of a fund with target $\pi_*$ and risk-aversion $\gamma$

\[
\sqrt{1 - R^2} = \frac{\sqrt{3}|1 - \pi_*|}{6} \left( \frac{6}{\gamma \pi_* (1 - \pi_*)} \right)^{1/3} \varepsilon^{1/3} + O(\varepsilon)
\]

**Alpha is excess exposure minus average trading cost**

\[
\tilde{\alpha} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \mu(\pi_t - \pi_*) dt - \varepsilon \pi_t \frac{d\varphi_t}{\varphi_t} \right) = -\frac{3 \sigma^2}{\gamma} \left( \frac{\gamma \pi_* (1 - \pi_*)}{6} \right)^{4/3} \varepsilon^{2/3} + O(\varepsilon)
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**The Limits of Leverage**

**Alpha vs. Tracking Error**

**Theorem**

For the \( R^2 = \lim_{T \to \infty} \left( \frac{1}{T} \int_0^T \pi_t dt \right)^2 \) of a fund with target \( \pi^*_t \) and risk-aversion \( \gamma \)

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The Limits of Leverage

Alpha vs. Tracking Error

**Theorem**

For the $R^2 = \lim_{T \to \infty} \frac{\left(\frac{1}{T} \int_0^T \pi_t dt\right)^2}{\frac{1}{T} \int_0^T \pi_t^2 dt}$ of a fund with target $\pi_*$ and risk-aversion $\gamma$

\[
\sqrt{1 - R^2} = \frac{\sqrt{3}|1 - \pi_*|}{6} \left(\frac{6}{\gamma \pi_*(1 - \pi_*)}\right)^{1/3} \varepsilon^{1/3} + O(\varepsilon)
\]

*Alpha is excess exposure minus average trading cost*

\[
\bar{\alpha} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\mu(\pi_t - \pi_*) dt - \varepsilon \pi_t \frac{d\varphi_t}{\varphi_t}\right) = -\frac{3\sigma^2}{\gamma} \left(\frac{\gamma \pi_*(1 - \pi_*)}{6}\right)^{4/3} \varepsilon^{2/3} + O(\varepsilon)
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Relative Tracking Error

- Relative tracking error $\sqrt{1 - R^2}$ (y) against leverage (x).
- Relative tracking error better than $R^2$ for tracking quality.
- $R^2$ high even beyond the leverage multiplier. Risk without return.

$\mu = 8\%, \sigma = 16\%, \varepsilon = 0.01\%(bottom), 0.1\%, 1\%(top)$
Relative Tracking Error

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\[ \mu = 8\%, \sigma = 16\%, \varepsilon = 0.01\% (\text{bottom}), 0.1\%, 1\% (\text{top}) \]
Relative Tracking Error

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Sketch of Argument (1)

- Summarize holdings by risky/safe ratio $\zeta_t = \pi_t/(1 - \pi_t)$.
- For some $\lambda$, conjecture finite-horizon value of the form

$$E_s \left[ \int_s^T \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_s^T \pi_t \frac{d\varphi_t^\dagger}{\varphi_t} \right] = V(\zeta_s) + \lambda(T - s)$$

- $V(\zeta) + \lambda(T - s) + \int_s^T \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_s^T \pi_t \frac{d\varphi_t^\dagger}{\varphi_t}$ supermartingale:

$$V'(\zeta_t) d\zeta_t + \frac{1}{2} V''(\zeta_t) d\langle \zeta_t \rangle_t - \lambda dt + \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \pi_t \frac{d\varphi_t^\dagger}{\varphi_t}$$

$$= \left( \frac{\sigma^2}{2} \zeta_t^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1 + \zeta} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta}{\zeta + 1} \right)^2 - \lambda \right) dt + V'(\zeta_t) \zeta_t \sigma dB_t$$

$$+ V'(\zeta_t) \zeta_t (1 + \zeta_t) \frac{d\varphi_t^\dagger}{\varphi_t} + \left( \varepsilon \frac{\zeta_t}{1 + \zeta_t} - V'(\zeta_t) \zeta_t (1 + (1 - \varepsilon) \zeta_t) \right) \frac{d\varphi_t^\dagger}{\varphi_t}.$$

- $dt$ term nonpositive, and zero on $[\zeta_-, \zeta_+]$
- $d\varphi_t^\dagger, d\varphi_t^\downarrow$ terms nonpositive, and zero at $\zeta_-, \zeta_+$ respectively.
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\]

- \( V(\zeta) + \lambda(T - s) + \int_s^T \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_s^T \pi_t \frac{d\varphi_t}{\varphi_t} \) supermartingale:

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\]

\[
= \left( \frac{\sigma^2}{2} \zeta_t^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \left( \frac{\zeta_t}{1+\zeta_t} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta_t}{\zeta_t+1} \right)^2 - \lambda \right) \right) dt + V'(\zeta_t) \zeta_t \sigma dB_t
\]

\[
+ V'(\zeta_t) \zeta_t (1 + \zeta_t) \frac{d\varphi_t}{\varphi_t} + \left( \varepsilon \left( \frac{\zeta_t}{1+\zeta_t} - V'(\zeta_t) \zeta_t (1 + (1 - \varepsilon) \zeta_t) \right) \right) \frac{d\varphi_t}{\varphi_t}
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- \( dt \) term nonpositive, and zero on \([\zeta_-, \zeta_+]\)
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- \( V(\zeta) + \lambda(T - s) + \int_s^T \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \int_s^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} \) supermartingale:

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V'(\zeta_t) d\zeta_t + \frac{1}{2} V''(\zeta_t) d\langle \zeta_t \rangle_t - \lambda dt + \left( \mu \pi_t - \frac{\gamma \sigma^2}{2} \pi_t^2 \right) dt - \varepsilon \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t}
\]

\[
= \left( \frac{\sigma^2}{2} \zeta_t^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1 + \zeta} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta}{\zeta + 1} \right)^2 - \lambda \right) dt + V'(\zeta_t) \zeta_t \sigma dB_t
\]

\[
+ V'(\zeta_t) \zeta_t (1 + \zeta_t) \frac{d\varphi_t^\downarrow}{\varphi_t} + \left( \varepsilon \frac{\zeta_t}{1 + \zeta_t} - V'(\zeta_t) \zeta_t (1 + (1 - \varepsilon) \zeta_t) \right) \frac{d\varphi_t^\uparrow}{\varphi_t}.
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- \( dt \) term nonpositive, and zero on \([\zeta_-, \zeta_+]\).

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  \[ E_s \left[ \int_s^T \left( \mu\pi_t - \frac{\gamma\sigma^2}{2}\pi_t^2 \right) dt - \varepsilon \int_s^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t} \right] = V(\zeta_s) + \lambda(T - s) \]

- $V(\zeta) + \lambda(T - s) + \int_s^T \left( \mu\pi_t - \frac{\gamma\sigma^2}{2}\pi_t^2 \right) dt - \varepsilon \int_s^T \pi_t \frac{d\varphi_t^\downarrow}{\varphi_t}$ supermartingale:
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  V'(\zeta_t)d\zeta_t + \frac{1}{2}V''(\zeta_t)d\langle\zeta\rangle_t - \lambda dt + \left( \mu\pi_t - \frac{\gamma\sigma^2}{2}\pi_t^2 \right) dt - \varepsilon\pi_t \frac{d\varphi_t^\downarrow}{\varphi_t}
  \]
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  = \left( \frac{\sigma^2}{2} \zeta_t^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1+\zeta} - \frac{\gamma\sigma^2}{2} \left( \frac{\zeta}{\zeta+1} \right)^2 - \lambda \right) dt + V'(\zeta_t)\zeta_t \sigma d\zeta_t
  \]
  \[
  + V'(\zeta_t)\zeta_t (1 + \zeta_t) \frac{d\varphi_t^\uparrow}{\varphi_t} + \left( \varepsilon \frac{\zeta_t}{1+\zeta_t} - V'(\zeta_t)\zeta_t (1 + (1 - \varepsilon)\zeta_t) \right) \frac{d\varphi_t^\downarrow}{\varphi_t}.
  \]
- $dt$ term nonpositive, and zero on $[\zeta_-, \zeta_+]$
- $d\varphi_t^\uparrow, d\varphi_t^\downarrow$ terms nonpositive, and zero at $\zeta_-, \zeta_+$ respectively.
Sketch of Argument (2)

- Hamilton-Jacobi-Bellman equation

\[
\frac{\sigma^2}{2} \zeta_t^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1 + \zeta} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta}{\zeta + 1} \right)^2 - \lambda = 0
\]

- Take derivative: second-order equation for \( W = -V' \). No \( \lambda \).

\[
\frac{\sigma^2}{2} \zeta^2 W''(\zeta) + (\sigma^2 + \mu) \zeta W'(\zeta) + \mu W(\zeta) - \frac{1}{(1 + \zeta)^2} \left( \mu - \frac{\gamma \sigma^2 \zeta}{1 + \zeta} \right) = 0
\]

- Four unknowns \((c_1, c_2, \zeta_-, \zeta_+)\), two boundary conditions.
- Smooth pasting conditions at \( \zeta_- \) and \( \zeta_+ \).
- Now four equations and four unknowns. One solution.
- Recover \( \lambda \) from first equation.
Sketch of Argument (2)

- Hamilton-Jacobi-Bellman equation

\[
\frac{\sigma^2}{2} \zeta_t^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1 + \zeta} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta}{\zeta + 1} \right)^2 - \lambda = 0
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  \frac{\sigma^2}{2} \zeta^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1 + \zeta} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta}{\zeta + 1} \right)^2 - \lambda = 0
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\]

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\[
\frac{\sigma^2}{2} \zeta^2 \mathcal{W}''(\zeta) + (\sigma^2 + \mu) \zeta \mathcal{W}'(\zeta) + \mu \mathcal{W}(\zeta) - \frac{1}{(1 + \zeta)^2} \left( \mu - \frac{\gamma \sigma^2 \zeta}{1 + \zeta} \right) = 0
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- Hamilton-Jacobi-Bellman equation

\[
\frac{\sigma^2}{2} \zeta^2 V''(\zeta_t) + \mu \zeta_t V'(\zeta_t) + \mu \frac{\zeta}{1 + \zeta} - \frac{\gamma \sigma^2}{2} \left( \frac{\zeta}{\zeta + 1} \right)^2 - \lambda = 0
\]

- Take derivative: second-order equation for \( W = -V' \). No \( \lambda \).

\[
\frac{\sigma^2}{2} \zeta^2 W''(\zeta) + (\sigma^2 + \mu) \zeta W'(\zeta) + \mu W(\zeta) - \frac{1}{(1 + \zeta)^2} \left( \mu - \frac{\gamma \sigma^2 \zeta}{1 + \zeta} \right) = 0
\]

- Four unknowns \((c_1, c_2, \zeta_-, \zeta_+)\), two boundary conditions.
- Smooth pasting conditions at \( \zeta_- \) and \( \zeta_+ \).
- Now four equations and four unknowns. One solution.
- Recover \( \lambda \) from first equation.
Conclusion

• Maximize average return for fixed volatility. Without frictions, usual frontier.

• Trading costs!

• Leverage cannot increase expected returns indefinitely. Maximum leverage multiplier finite.

• Multiplier increases with liquidity and returns. Decreases with volatility.

• Between two assets with equal Sharpe ratio, more volatility better. Superior frontier.

• Embedded leverage without constraints, but with trading costs.

• Optimal tradeoff between alpha and tracking error.
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Thank You!

Questions?