Optimal Investment with Transaction Costs and Stochastic Volatility

Ronnie Sircar

Department of Operations Research and Financial Engineering
Princeton University
http://www.princeton.edu/~sircar

Joint with: Maxim Bichuch (WPI)
Overview I

- **Perturbation tools** are classical in tackling complicated problems by reducing them to a sequence of simpler ones.
- In financial applications, they are prevalent in **option pricing** (or smile calibration) in various regimes (*references omitted*):
  - Contract asymptotics: small or large $K$ or $T$;
  - Model asymptotics: stochastic volatility time scales; parametrix-type expansions, etc.
- For the nonlinear **portfolio optimization**, how can these methods help to compute (or approximate) value functions and optimal policies?
- **Outlook**: viewing an **incomplete market problem** as a **perturbation** around a well-understood (complete markets) problem.
- Here we analyze **multiscale stochastic volatility (MS-SV) + transaction costs**. “**Vega-Gamma**” relationship between the Greeks.
Overview I

▶ Perturbation tools are classical in tackling complicated problems by reducing them to a sequence of simpler ones.
▶ In financial applications, they are prevalent in option pricing (or smile calibration) in various regimes (*references omitted*):
  ▶ **Contract asymptotics**: small or large $K$ or $T$;
  ▶ **Model asymptotics**: stochastic volatility time scales; parametrix-type expansions, etc.
▶ For the nonlinear portfolio optimization, how can these methods help to compute (or approximate) value functions and optimal policies?
▶ **Outlook**: viewing an incomplete market problem as a perturbation around a well-understood (complete markets) problem.
▶ Here we analyze multiscale stochastic volatility (MS-SV) + transaction costs. “Vega-Gamma” relationship between the Greeks.
Overview I

➤ **Perturbation tools** are classical in tackling complicated problems by reducing them to a sequence of simpler ones.

➤ In financial applications, they are prevalent in **option pricing** (or smile calibration) in various regimes (*references omitted*):
  - **Contract asymptotics:** small or large $K$ or $T$;
  - **Model asymptotics:** stochastic volatility time scales; parametrix-type expansions, etc.

➤ For the nonlinear **portfolio optimization**, how can these methods help to compute (or approximate) value functions and optimal policies?

➤ **Outlook:** viewing an incomplete market problem as a perturbation around a well-understood (complete markets) problem.

➤ Here we analyze **multiscale stochastic volatility (MS-SV) + transaction costs. “Vega-Gamma” relationship between the Greeks.**
Overview I

- **Perturbation tools** are classical in tackling complicated problems by reducing them to a sequence of simpler ones.
- In financial applications, they are prevalent in option pricing (or smile calibration) in various regimes (*references omitted*):
  - **Contract asymptotics**: small or large $K$ or $T$;
  - **Model asymptotics**: stochastic volatility time scales; parametrix-type expansions, etc.
- For the nonlinear **portfolio optimization**, how can these methods help to compute (or approximate) value functions and optimal policies?
- **Outlook**: viewing an **incomplete market problem** as a **perturbation** around a well-understood (complete markets) problem.
- Here we analyze multiscale stochastic volatility (MS-SV) + transaction costs. “Vega-Gamma” relationship between the Greeks.
Overview I

- **Perturbation tools** are classical in tackling complicated problems by reducing them to a sequence of simpler ones.
- In financial applications, they are prevalent in **option pricing** (or smile calibration) in various regimes (*references omitted*):
  - **Contract asymptotics**: small or large $K$ or $T$;
  - **Model asymptotics**: stochastic volatility time scales; parametrix-type expansions, etc.
- For the nonlinear **portfolio optimization**, how can these methods help to compute (or approximate) value functions and optimal policies?
- **Outlook**: viewing an **incomplete market problem** as a **perturbation** around a well-understood (complete markets) problem.
- Here we analyze **multiscale stochastic volatility (MS-SV)** + **transaction costs**. “Vega-Gamma” relationship between the **Greeks**.
Two much-studied frictions in financial markets: uncertain changing volatility and transaction costs. How do these jointly impact portfolio optimization? – the Merton problem of how to allocate between risk & riskless assets to maximize expected utility.

Have been studied separately:

- Transaction costs & constant volatility: often by asymptotic expansions in small costs parameter.
- Stochastic volatility & zero costs: some explicit solutions for special models; some asymptotics for multiscale volatility framework.

Here: separation of time scale approximations for the volatility friction, with fixed transaction cost.

Perturbation analysis of an infinite horizon free-boundary eigenvalue problem.
Transaction Costs & Stochastic Volatility

- Two much-studied frictions in financial markets: uncertain changing volatility and transaction costs.
- How do these jointly impact portfolio optimization? – the Merton problem of how to allocate between risk & riskless assets to maximize expected utility.
- Have been studied separately:
  - Transaction costs & constant volatility: often by asymptotic expansions in small costs parameter.
  - Stochastic volatility & zero costs: some explicit solutions for special models; some asymptotics for multiscale volatility framework.
- Here: separation of time scale approximations for the volatility friction, with fixed transaction cost.
- Perturbation analysis of an infinite horizon free-boundary eigenvalue problem.
Transaction Costs & Stochastic Volatility

- Two much-studied frictions in financial markets: uncertain changing volatility and transaction costs.

- How do these *jointly* impact portfolio optimization? – the Merton problem of how to allocate between risk & riskless assets to maximize expected utility.

- Have been studied separately:
  - **Transaction costs & constant volatility**: often by asymptotic expansions in small costs parameter.
  - **Stochastic volatility & zero costs**: some explicit solutions for special models; some asymptotics for multiscale volatility framework.

- Here: separation of time scale approximations for the volatility friction, with fixed transaction cost.

- Perturbation analysis of an infinite horizon free-boundary eigenvalue problem.
Two much-studied frictions in financial markets: uncertain changing volatility and transaction costs.

How do these jointly impact portfolio optimization? – the Merton problem of how to allocate between risk & riskless assets to maximize expected utility.

Have been studied separately:

- Transaction costs & constant volatility: often by asymptotic expansions in small costs parameter.
- Stochastic volatility & zero costs: some explicit solutions for special models; some asymptotics for multiscale volatility framework.

Here: separation of time scale approximations for the volatility friction, with fixed transaction cost.

Perturbation analysis of an infinite horizon free-boundary eigenvalue problem.
Transaction Costs & Stochastic Volatility

- Two much-studied frictions in financial markets: uncertain changing volatility and transaction costs.
- How do these *jointly* impact portfolio optimization? – the Merton problem of how to allocate between risk & riskless assets to maximize expected utility.
- Have been studied separately:
  - Transaction costs & constant volatility: often by asymptotic expansions in small costs parameter.
  - Stochastic volatility & zero costs: some explicit solutions for special models; some asymptotics for multiscale volatility framework.
- Here: separation of time scale approximations for the volatility friction, with fixed transaction cost.
- Perturbation analysis of an infinite horizon free-boundary eigenvalue problem.
Transaction Costs Background: Constant (or local) Volatility

- **Merton problem**: Magill & Constantinides (1976); Davis & Norman (1990); Dumas & Luciano (1991). Solution characterized by a no trade (NT) region.


- **Asymptotics for small costs**: Shreve & Soner (1994); Whalley & Wilmott (1997); Jancek & Shreve (2004); Soner & Touzi (2012).

- **Shadow prices**: Gerhold, Guasoni, Muhle-Karbe & Schachermayer (2011). **Multiple stocks**: Bichuch & Shreve (2011); Possamai, Soner & Touzi (2012).


Here (to begin) take volatility in the family of fast ergodic diffusions:

\[
\frac{dS_t}{S_t} = (\mu + r) \, dt + f(Z_t) \, dB^1_t, \\
dZ_t = \frac{1}{\varepsilon} \alpha(Z_t) \, dt + \frac{1}{\sqrt{\varepsilon}} \beta(Z_t) \, dB^2_t,
\]

with \( d \langle B^1, B^2 \rangle_t = \rho \, dt \), and typical \( \rho \in (-1, 1) \) is negative.

Say vol is fast mean-reverting if \( \varepsilon > 0 \) is small, and \( Z \) is an ergodic process with a unique invariant distribution \( \Phi \) that is independent of \( \varepsilon \).
Fast Mean-Reverting Stochastic Volatility: $\kappa = 1/\varepsilon$

Figure: Simulated CIR Stochastic Volatility (Heston)
Investment Problem

- Investor chooses a policy consisting of two adapted processes $L$ and $M$ that are nondecreasing.

- The control $L_t$ represents the cumulative dollar value of stock purchased up to time $t$, while $M_t$ is the cumulative dollar value of stock sold.

- Then, the wealth $X$ invested in the money market account and the wealth $Y$ invested in the stock follow

\[
\begin{align*}
    dX_t &= rX_t \, dt - dL_t + (1 - \lambda) \, dM_t, \\
    dY_t &= (\mu + r) \, Y_t \, dt + f(Z_t) \, Y_t \, dB^1_t + dL_t - dM_t.
\end{align*}
\]

The constant $\lambda \in (0, 1)$ represents the proportional transaction costs for selling the stock.
Optimization Problem

▶ Power utility functions:

\[ U(w) = \frac{w^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1. \]

▶ Problem is to maximize long term growth of expected utility:

\[
\lim_{T \to \infty} \inf T \sup (L, M) \in A(x, y, z) \log U^{-1} \left( \mathbb{E}_0^{X, Y, Z} [U(X_T + Y_T)] \right).
\]

▶ Value function for utility maximization at a finite time horizon \( T \):

\[
V(t, X_t, Y_t, Z_t) = \sup (L, M) \in A(x, y, z) \mathbb{E}_t^{X_t, Y_t, Z_t} [U(X_T + Y_T)].
\]

Standard ansatz:

\[
V(t, x, y, z) = x^{1-\gamma} v^{\lambda, \varepsilon} (\zeta, z) e^{(1-\gamma)(r+\delta\varepsilon)(T-t)}, \quad \zeta = \frac{y}{x},
\]

where \( \delta\varepsilon \) is the effective safe rate to be found.
Free Boundary Formulation

Then one looks for a solution to the associated variational inequality with a no trade region $\zeta \in (\ell^\varepsilon(z), u^\varepsilon(z))$, and $C^2$ continuity conditions at the buy and sell boundaries.

In the NT region:

$$\left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + (\mathcal{L}_2 - (1 - \gamma) \delta^\varepsilon) \right) \nu^{\lambda,\varepsilon} = 0, \quad \zeta \in (\ell^\varepsilon(z), u^\varepsilon(z))$$

where we have $\mathcal{L}_0 = \frac{1}{2} \beta^2(z) \partial_{zz}^2 + \alpha(z) \partial_z$

$$\mathcal{L}_1 = \rho f(z) \beta(z) \zeta \partial_z^2, \quad \mathcal{L}_2 = \frac{1}{2} f^2(z) \zeta^2 \partial_{\zeta \zeta} + \mu \zeta \partial_\zeta.$$

$\mathcal{L}_0$ is the generator of $Z$ with $\varepsilon = 1$; $\mathcal{L}_1$ contains the correlation; $\mathcal{L}_2$ is the base operator with volatility $f(z)$. 
Boundary Conditions

At the lower buy boundary

\[(1 + \ell^\varepsilon(z)) \nu^{\lambda,\varepsilon}_{\xi} (\ell^\varepsilon(z)) - (1 - \gamma) \nu^{\lambda,\varepsilon}(\ell^\varepsilon(z)) = 0,\]

\[(1 + \ell^\varepsilon(z)) \nu^{\lambda,\varepsilon}_{\xi\xi} (\ell^\varepsilon(z)) + \gamma \nu^{\lambda,\varepsilon}_\xi (\ell^\varepsilon(z)) = 0,\]

and at the upper sell boundary

\[\left(\frac{1}{1 - \lambda} + u^\varepsilon(z)\right) \nu^{\lambda,\varepsilon}_{\xi} (u^\varepsilon(z)) - (1 - \gamma) \nu^{\lambda,\varepsilon}(u^\varepsilon(z)) = 0,\]

\[\left(\frac{1}{1 - \lambda} + u^\varepsilon(z)\right) \nu^{\lambda,\varepsilon}_{\xi\xi} (u^\varepsilon(z)) + \gamma \nu^{\lambda,\varepsilon}_\xi (u^\varepsilon(z)) = 0.\]

These are homogeneous equations with homogeneous boundary conditions and so zero is a solution.

However $\delta^\varepsilon$ is also to be determined: it is an eigenvalue found to exclude only the trivial solution, and give the optimal long run growth rate.
Fast Mean-Reverting Asymptotics

Look for an expansion for the value function

$$v^{\lambda,\varepsilon} = v^{\lambda,0} + \sqrt{\varepsilon} v^{\lambda,1} + \varepsilon v^{\lambda,2} + \cdots,$$

as well as for the free boundaries

$$\ell^{\varepsilon} = \ell_0 + \sqrt{\varepsilon} \ell_1 + \varepsilon \ell_2 + \cdots, \quad u^{\varepsilon} = u_0 + \sqrt{\varepsilon} u_1 + \varepsilon u_2 + \cdots,$$

and the optimal excess growth rate

$$\delta^{\varepsilon} = \delta_0 + \sqrt{\varepsilon} \delta_1 + \cdots,$$

which are asymptotic as $\varepsilon \downarrow 0$.

The principal term $v^{\lambda,0}(\zeta)$ solves the constant volatility problem with square averaged volatility $\overline{\sigma}$:

$$\overline{\sigma}^2 = \langle f^2 \rangle = \int f^2 \Phi.$$
Principal Term

In the principal NT region:

\[ \frac{1}{2} \sigma^2 \zeta^2 \nu_{\zeta \zeta}^{\lambda,0} + \mu \zeta \nu_{\zeta}^{\lambda,0} - (1 - \gamma) \delta_0 \nu^{\lambda,0} = 0, \quad 0 < \ell_0 \leq \zeta \leq u_0, \]

with boundary conditions \((1 = 1\text{st deriv.}, \ 2 = 2\text{nd deriv.})\)

\[
B_1^0 \nu^{\lambda,0} := (1 + \ell_0) \nu_{\zeta}^{\lambda,0}(\ell_0) - (1 - \gamma) \nu^{\lambda,0}(\ell_0) = 0, \\
B_2^0 \nu^{\lambda,0} := (1 + \ell_0) \nu_{\zeta \zeta}^{\lambda,0}(\ell_0) + \gamma \nu_{\zeta}^{\lambda,0}(\ell_0) = 0,
\]

at the lower \textbf{buy} boundary \(\ell_0\) and

\[
S_1^0 \nu^{\lambda,0} := \left( \frac{1}{1 - \lambda} + u_0 \right) \nu_{\zeta}^{\lambda,0}(u_0) - (1 - \gamma) \nu^{\lambda,0}(u_0) = 0, \\
S_2^0 \nu^{\lambda,0} := \left( \frac{1}{1 - \lambda} + u_0 \right) \nu_{\zeta \zeta}^{\lambda,0}(u_0) + \gamma \nu_{\zeta}^{\lambda,0}(u_0) = 0.
\]

at the upper \textbf{sell} boundary \(u_0\).
Principal Solution

- Take the two independent solutions \( v_\pm(\zeta) \) of the homogeneous ODE, then \( v^{\lambda,0} = c_+ v_+ + c_- v_- \).

- From the first boundary conditions \( M \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = 0 \):

\[
M = \begin{pmatrix}
    v_+(\ell_0) - k_\ell v'_+(\ell_0) & v_-(\ell_0) - k_\ell v'_-(\ell_0) \\
    v_+(u_0) - k_u v'_+(u_0) & v_-(u_0) - k_u v'_-(u_0)
\end{pmatrix}.
\]

For a non-trivial solution, the algebraic equation \(|M| = 0\) determines \( \delta_0 \).

- The free boundaries \( \ell_0 \) and \( u_0 \) are determined by the remaining boundary conditions.

- Value function \( v^{\lambda,0} \) determined up to a multiplicative constant along the eigenvector. Solutions are either \( v_\pm(\zeta) = \zeta^{\theta \pm} \) or

\[
v_+(\zeta) = \zeta^{\theta_r} \cos(\theta_i \log \zeta), \quad v_-(\zeta) = \zeta^{\theta_r} \sin(\theta_i \log \zeta),
\]

depending on roots of \( \frac{1}{2} \sigma^2 \theta^2 + (\mu - \frac{1}{2} \sigma^2) \theta - (1 - \gamma) \delta_0 = 0 \).
Finding $\nu^{\lambda,1}$ and $\delta_1$

Define

$$L_{NT} = \frac{1}{2} \sigma^2 D_2 + \mu D_1 - (1 - \gamma) \delta_0, \quad D_k = \zeta_k \frac{\partial^k}{\partial \zeta^k}.$$ 

So we had $L_{NT} \nu^{\lambda,0} = 0$ in the NT region.

The asymptotic analysis and Fredholm alternative leads to

$$L_{NT} \nu^{\lambda,1} = -V_3 D_1 D_2 \nu^{\lambda,0} + (1 - \gamma) \delta_1 \nu^{\lambda,0}, \quad V_3 = -\frac{1}{2} \rho \langle \beta f \phi' \rangle,$$

where $L_0 \phi = f^2 - \sigma^2$.

So $V_3$ contains the effect of the stock-vol correlation.

This ODE holds on $(\ell_0, u_0)$ with Neumann boundary conditions $B^0_1 \nu^{\lambda,1} = 0 = S^0_1 \nu^{\lambda,1}$. 


Finding $v^{\lambda,1}$ and $\delta_1$ ctd.

Let $w$ be the solution to the adjoint problem $L_{NT}^* w = 0$ with appropriate adjoint boundary conditions. In fact, $w(\zeta) = \zeta^{k-2} v^{\lambda,0}(\zeta)$, where $k = \frac{\mu}{2\sigma^2}$.

Then inner product $v^{\lambda,1}$ equation with $w$ to obtain:

$$\delta_1 = \frac{V_3}{(1 - \gamma)} \frac{\int_{\ell_0}^{u_0} w D_1 D_2 v^{\lambda,0} d\zeta}{\int_{\ell_0}^{u_0} w v^{\lambda,0} d\zeta}.$$ 

Can be computed explicitly.

Correction $\delta_1$ to effective safe rate already determined by $v^{\lambda,0}$. It is chosen so that the $v^{\lambda,1}$ equation with the first pair of boundary conditions has a solution (Fredholm solvability condition).
First Correction to Value and Boundaries

We obtain $v^{\lambda,1}$ up to an addition of a multiple of $v^{\lambda,0}$:

$$v^{\lambda,1} = A_+ (\zeta) v_+ (\zeta) + A_- (\zeta) v_- (\zeta) + C_+ v_+ (\zeta) + \xi v^{\lambda,0} (\zeta),$$

for any $\xi$. $A_\pm$ determined from Wronskian calculations.

The remaining pair of boundary conditions, expanded to 1st order give:

$$\ell_1 = - \left( \frac{(1 + \ell_0) v^{\lambda,1}_{\zeta\zeta} (\ell_0) + \gamma v^{\lambda,1}_{\zeta} (\ell_0)}{(1 + \ell_0) v^{\lambda,0}_{\zeta\zeta\zeta} (\ell_0) + (1 + \gamma) v^{\lambda,0}_{\zeta\zeta} (\ell_0)} \right),$$

$$u_1 = - \left( \frac{\left( \frac{1}{1 - \lambda} + u_0 \right) v^{\lambda,1}_{\zeta\zeta} (u_0) + \gamma v^{\lambda,1}_{\zeta} (u_0)}{\left( \frac{1}{1 - \lambda} + u_0 \right) v^{\lambda,0}_{\zeta\zeta\zeta} (u_0) + (1 + \gamma) v^{\lambda,0}_{\zeta\zeta} (u_0)} \right),$$

independent of $\xi$. 
Figure: Safe rates $\hat{\delta}_0$ and $\hat{\delta}_0 + \sqrt{\varepsilon}\hat{\delta}_1$ in the fast-scale stochastic volatility with negative correlation.
Figure: Boundaries $\ell_0, u_0$ and $\ell_0 + \sqrt{\varepsilon} l_1, u_0 + \sqrt{\varepsilon} u_1$ in the fast-scale stochastic volatility.
Slow Volatility Approximation

▶ A different approximation arises if instead we parametrize volatility as slowly fluctuating compared with the stock:

\[ \sigma_t = f(Z_t) \]

where

\[ dZ_t = \varepsilon \alpha(Z_t) \, dt + \sqrt{\varepsilon} \beta(Z_t) \, dB_t^2, \quad d\langle B^1, B^2 \rangle_t = \rho \, dt. \]

Here \( Z \) does not need to be ergodic.

▶ Then the key difference is that in the principal terms, volatility is frozen at \( f(z) \) instead of square-averaged. \( v^{\lambda,0} = v_0(\zeta; f(z)) \) is as before with the constant volatility \( f(z) \) instead of \( \bar{\sigma} \):

\[ \mathcal{L}_{NT} v^{\lambda,0} = 0, \quad \mathcal{L}_{NT} = \frac{1}{2} f(z)^2 D_2 + \mu D_1 - (1 - \gamma) \delta_0 \cdot . \]

▶ In the correction, we essentially replace

\[ D_1 D_2 \mapsto D_1 \partial_z = \zeta \partial_{\zeta}^2. \]

That is the correction satisfies

\[ \mathcal{L}_{NT} v^{\lambda,1} = -\rho f(z) \beta(z) D_1 \partial_z v^{\lambda,0} + (1 - \gamma) \delta_1 v^{\lambda,0}. \]
Slow Volatility Approximation

A different approximation arises if instead we parametrize volatility as slowly fluctuating compared with the stock: 
\[ \sigma_t = f(Z_t) \]
where

\[ dZ_t = \varepsilon \alpha(Z_t) \, dt + \sqrt{\varepsilon} \beta(Z_t) \, dB_t^2, \quad d\left\langle B^1, B^2 \right\rangle_t = \rho \, dt. \]

Here \( Z \) does not need to be ergodic.

Then the key difference is that in the principal terms, volatility is \textit{frozen} at \( f(z) \) instead of square-averaged. \( \nu^{\lambda, 0} = \nu_0(\zeta; f(z)) \) is as before with the constant volatility \( f(z) \) instead of \( \bar{\sigma} \):

\[ \mathcal{L}_{NT} \nu^{\lambda, 0} = 0, \quad \mathcal{L}_{NT} = \frac{1}{2} f(z)^2 D_2 + \mu D_1 - (1 - \gamma) \delta_0 \cdot. \]

In the correction, we essentially replace

\[ D_1 D_2 \mapsto D_1 \partial_z = \zeta \partial_{\zeta}^2. \]

That is the correction satisfies

\[ \mathcal{L}_{NT} \nu^{\lambda, 1} = -\rho f(z) \beta(z) D_1 \partial_z \nu^{\lambda, 0} + (1 - \gamma) \delta_1 \nu^{\lambda, 0}. \]
Slow Volatility Approximation

▶ A different approximation arises if instead we parametrize volatility as slowly fluctuating compared with the stock:
\( \sigma_t = f(Z_t) \) where
\[
dZ_t = \varepsilon \alpha(Z_t) \, dt + \sqrt{\varepsilon} \beta(Z_t) \, dB_t^2, \quad d\langle B^1, B^2 \rangle_t = \rho \, dt.
\]
Here \( Z \) does not need to be ergodic.

▶ Then the key difference is that in the principal terms, volatility is frozen at \( f(z) \) instead of square-averaged.
\( v^{\lambda,0} = v_0(\zeta; f(z)) \) is as before with the constant volatility \( f(z) \) instead of \( \bar{\sigma} \):
\[
\mathcal{L}_{\text{NT}} v^{\lambda,0} = 0, \quad \mathcal{L}_{\text{NT}} = \frac{1}{2} f(z)^2 D_2 + \mu D_1 - (1 - \gamma) \delta_0 \cdot .
\]

▶ In the correction, we essentially replace
\[
D_1 D_2 \mapsto D_1 \partial_z = \zeta \partial^2_{\zeta z}.
\]
That is the correction satisfies
\[
\mathcal{L}_{\text{NT}} v^{\lambda,1} = -\rho f(z) \beta(z) D_1 \partial_z v^{\lambda,0} + (1 - \gamma) \delta_1 v^{\lambda,0}.
\]
Computing $\nu_{1}^{\lambda,1}$ and hence $\ell_{1}$ and $u_{1}$ requires $\partial_{z}\nu_{1}^{\lambda,0}(\zeta; z)$ which involves finding $\partial_{\sigma}\delta_{0} =: \delta_{0,\sigma}$.

In other situations, to obtain explicit formulas for the correction terms, it is helpful to write (for the constant volatility value function) the “Vega” $\partial_{\sigma}$ in terms of the “Gamma” $D_{2}$.

European Option Pricing with maturity $T < \infty$ at time $t < T$:

$$D_{k} = S_{k}^{k} \frac{\partial^{k}}{\partial S^{k}}, \quad \partial_{\sigma} P_{BS} = \sigma(T - t)D_{2}P_{BS}.$$ 

Long convexity $\Rightarrow$ Long Volatility.
Vega-Gamma Relationship

- Computing $\nu^{\lambda,1}$ and hence $\ell_1$ and $u_1$ requires $\partial_z \nu^{\lambda,0}(\zeta; z)$ which involves finding $\partial_\sigma \delta_0 =: \delta_{0,\sigma}$.

- In other situations, to obtain explicit formulas for the correction terms, it is helpful to write (for the constant volatility value function) the “Vega” $\partial_\sigma$ in terms of the “Gamma” $D_2$.

- **European Option Pricing** with maturity $T < \infty$ at time $t < T$:

  $$D_k = S_k \frac{\partial^k}{\partial S^k}, \quad \partial_\sigma P_{BS} = \sigma (T - t) D_2 P_{BS}.$$  

  *Long convexity $\Rightarrow$ Long Volatility.*
Vega-Gamma Relationship

- Computing $\nu^{\lambda,1}$ and hence $\ell_1$ and $u_1$ requires $\partial_z \nu^{\lambda,0}(\zeta; z)$ which involves finding $\partial_\sigma \delta_0 =: \delta_{0,\sigma}$.

- In other situations, to obtain explicit formulas for the correction terms, it is helpful to write (for the constant volatility value function) the “Vega” $\partial_\sigma$ in terms of the “Gamma” $D_2$.

- **European Option Pricing** with maturity $T < \infty$ at time $t < T$:

  $$D_k = S^k \frac{\partial^k}{\partial S^k}, \quad \partial_\sigma P_{BS} = \sigma (T - t) D_2 P_{BS}.$$ 

  *Long convexity $\Rightarrow$ Long Volatility.*
Vega-Gamma Relationship for Merton Problem

Let $M(t, x; \lambda_s)$ be the (frictionless) Merton value function corresponding to GBM with Sharpe ratio $\lambda_s$ where $x$ is wealth:

$$M_t - \frac{1}{2} \lambda_s^2 \frac{M_x^2}{M_{xx}} = 0, \quad M(T, x) = U(x).$$

Let $R(t, x; \lambda_s)$ be the risk-tolerance function

$$R = -\frac{M_x}{M_{xx}} \quad \text{and} \quad D_k = R(t, x)^k \frac{\partial^k}{\partial x^k}.$$

Can write

$$M_t + \frac{1}{2} \lambda_s^2 D_2 M + \lambda_s^2 D_1 M = 0.$$

Then (FSZ 2012):

$$\frac{\partial M}{\partial \lambda_s} = -(T - t) \lambda_s D_2 M.$$

Follows from Black’s (fast diffusion) equation for $R$:

$$R_t + \frac{1}{2} R^2 R_{xx} = 0.$$
Vega-Gamma Relationship for Merton Problem

Let $M(t, x; \lambda_s)$ be the (frictionless) Merton value function corresponding to GBM with Sharpe ratio $\lambda_s$ where $x$ is wealth:

$$M_t - \frac{1}{2} \lambda_s^2 \frac{M_x^2}{M_{xx}} = 0, \quad M(T, x) = U(x).$$

Let $R(t, x; \lambda_s)$ be the risk-tolerance function

$$R = -\frac{M_x}{M_{xx}} \quad \text{and} \quad D_k = R(t, x)^k \frac{\partial^k}{\partial x^k}.$$ 

Can write $M_t + \frac{1}{2} \lambda_s^2 D_2 M + \lambda_s^2 D_1 M = 0.$

Then (FSZ 2012):

$$\frac{\partial M}{\partial \lambda_s} = -(T - t) \lambda_s D_2 M.$$

Follows from Black’s (fast diffusion) equation for $R$:

$$R_t + \frac{1}{2} R^2 R_{xx} = 0.$$
Vega-Gamma Relationship for Merton Problem

Let $M(t, x; \lambda_s)$ be the (frictionless) Merton value function corresponding to GBM with Sharpe ratio $\lambda_s$ where $x$ is wealth:

$$M_t - \frac{1}{2} \lambda_s^2 \frac{M_x^2}{M_{xx}} = 0, \quad M(T, x) = U(x).$$

Let $R(t, x; \lambda_s)$ be the risk-tolerance function

$$R = -\frac{M_x}{M_{xx}} \quad \text{and} \quad D_k = R(t, x)^k \frac{\partial^k}{\partial x^k}.$$

Can write $M_t + \frac{1}{2} \lambda_s^2 D_2 M + \lambda_s^2 D_1 M = 0$.

Then (FSZ 2012):

$$\frac{\partial M}{\partial \lambda_s} = -(T - t) \lambda_s D_2 M.$$

Follows from Black’s (fast diffusion) equation for $R$:

$$R_t + \frac{1}{2} R^2 R_{xx} = 0.$$
When $T < \infty$, all that is happening is we have an operator

$$\mathcal{L} = \partial_t + \frac{1}{2}\sigma^2 D_2 + \mu D_1 - \nu,$$

where $D_1$ and $D_2$ commute (obvious for Black-Scholes option pricing, less so for Merton optimal control).

Then the value function or price solves $\mathcal{L}P = 0$ and its Vega $\nu = \partial_\sigma P$ solves the inhomogeneous equation

$$\mathcal{L}\nu = -\sigma D_2 P.$$

But $D_2 P$ solves the homogeneous equation too, so one solution is $\nu = (T - t)\sigma D_2 P$.

If there are no boundary conditions (or Vega is zero on the boundaries) then that is it (true for European options, not true for barrier options).
Derivation

When \( T < \infty \), all that is happening is we have an operator

\[
\mathcal{L} = \partial_t + \frac{1}{2} \sigma^2 D_2 + \mu D_1 - \nu, \]

where \( D_1 \) and \( D_2 \) commute (obvious for Black-Scholes option pricing, less so for Merton optimal control).

Then the value function or price solves \( \mathcal{L}P = 0 \) and its Vega \( \nu = \partial_\sigma P \) solves the inhomogeneous equation

\[
\mathcal{L}\nu = -\sigma D_2 P. \]

But \( D_2 P \) solves the homogeneous equation too, so one solution is \( \nu = (T - t)\sigma D_2 P \).

If there are no boundary conditions (or Vega is zero on the boundaries) then that is it (true for European options, not true for barrier options).
Derivation

When \( T < \infty \), all that is happening is we have an operator

\[
\mathcal{L} = \partial_t + \frac{1}{2} \sigma^2 D_2 + \mu D_1 - \nu 
\]

where \( D_1 \) and \( D_2 \) commute (obvious for Black-Scholes option pricing, less so for Merton optimal control).

Then the value function or price solves \( \mathcal{L}P = 0 \) and its Vega \( \nu = \partial_\sigma P \) solves the inhomogeneous equation

\[
\mathcal{L}\nu = -\sigma D_2 P 
\]

But \( D_2 P \) solves the homogeneous equation too, so one solution is \( \nu = (T - t)\sigma D_2 P \).

If there are no boundary conditions (or Vega is zero on the boundaries) then that is it (true for European options, not true for barrier options).
Vega-Gamma for Perpetuals?

▸ When \( T = \infty \), there is no \( \partial_t \), so inhomogeneous ODE (with source solving the homogeneous equation) brings a secular term.

▸ For the perpetual American put, we have

\[
\nu = (A \log S + B) D_2 P_{BS}(S)
\]

where \( A \) is known and \( B \) has to be fixed by computing Vega and Gamma at one value of \( S \).

▸ Returning to transaction costs, let \( \nu_0(\zeta; \sigma) \) be the constant volatility value function: \( \mathcal{L}_{NT} \nu_0 = 0 \).

▸ Differentiating ODE with respect to \( \sigma \) gives for \( \nu = \partial_\sigma \nu_0 \):

\[
\mathcal{L}_{NT} \nu = -\sigma D_2 \nu_0 + (1 - \gamma) \delta_{0,\sigma} \nu_0, \quad \delta_{0,\sigma} = \partial_\sigma \delta_0,
\]

with \( B_1^0 \nu = 0 = S_1^0 \nu \).
Vega-Gamma for Perpetuals?

- When \( T = \infty \), there is no \( \partial_t \), so inhomogeneous ODE (with source solving the homogeneous equation) brings a secular term.
- For the **perpetual American put**, we have

\[
\nu = (A \log S + B) D_2 P_{BS}(S)
\]

where \( A \) is known and \( B \) has to be fixed by computing Vega and Gamma at one value of \( S \).

- Returning to transaction costs, let \( \nu_0(\zeta; \sigma) \) be the constant volatility value function: \( \mathcal{L}_{NT} \nu_0 = 0 \).
- Differentiating ODE with respect to \( \sigma \) gives for \( \nu = \partial_\sigma \nu_0 \):

\[
\mathcal{L}_{NT} \nu = -\sigma D_2 \nu_0 + (1 - \gamma) \delta_{0,\sigma} \nu_0, \quad \delta_{0,\sigma} = \partial_\sigma \delta_0,
\]

with \( B_1^0 \nu = 0 = S_1^0 \nu \).
Vega-Gamma for Perpetuals?

- When \( T = \infty \), there is no \( \partial_t \), so inhomogeneous ODE (with source solving the homogeneous equation) brings a secular term.
- For the **perpetual American put**, we have

\[
\mathcal{V} = (A \log S + B) D_2 P_{BS}(S)
\]

where \( A \) is known and \( B \) has to be fixed by computing Vega and Gamma at one value of \( S \).

- Returning to transaction costs, let \( \nu_0(\zeta; \sigma) \) be the constant volatility value function: \( \mathcal{L}_{NT} \nu_0 = 0 \).
- Differentiating ODE with respect to \( \sigma \) gives for \( \mathcal{V} = \partial_\sigma \nu_0 \):

\[
\mathcal{L}_{NT} \mathcal{V} = -\sigma D_2 \nu_0 + (1 - \gamma) \delta_{0,\sigma} \nu_0, \quad \delta_{0,\sigma} = \partial_\sigma \delta_0,
\]

with \( B_1^0 \mathcal{V} = 0 = S_1^0 \mathcal{V} \).
Vega-Gamma for Perpetuals?

► When \( T = \infty \), there is no \( \partial_t \), so inhomogeneous ODE (with source solving the homogeneous equation) brings a secular term.

► For the **perpetual American put**, we have

\[
\mathcal{V} = (A \log S + B)D_2 P_{BS}(S)
\]

where \( A \) is known and \( B \) has to be fixed by computing Vega and Gamma at one value of \( S \).

► Returning to transaction costs, let \( \nu_0(\zeta; \sigma) \) be the constant volatility value function: \( \mathcal{L}_{NT} \nu_0 = 0 \).

► Differentiating ODE with respect to \( \sigma \) gives for \( \mathcal{V} = \partial_\sigma \nu_0 \):

\[
\mathcal{L}_{NT} \mathcal{V} = -\sigma D_2 \nu_0 + (1 - \gamma) \delta_{0,\sigma} \nu_0, \quad \delta_{0,\sigma} = \partial_\sigma \delta_0,
\]

with \( B_1^0 \mathcal{V} = 0 = S_1^0 \mathcal{V} \).
Vega for Transaction Costs

► Recall \( w(\zeta) = \zeta^{k-2}v_0 \) solves the adjoint problem \( \mathcal{L}_{NT}^* w = 0 \) with adjoint boundary conditions. Then the Fredholm solvability condition gives

\[
\delta_{0,\sigma} = \frac{\sigma \int_{\ell_0}^{u_0} wD_2 v_0 \, d\zeta}{(1 - \gamma) \int_{\ell_0}^{u_0} w v_0 \, d\zeta}.
\]

This is easily computed explicitly (and saves a lot of sensitive numerical differentiation).

► Finally (in the case of real roots)

\[
\mathcal{V}(\zeta) = (Q_+ v_+(\zeta) + Q_- v_-(\zeta)) \log \zeta + C_+^{(V)} v_+(\zeta) + \eta v_0(\zeta),
\]

and the appropriate \( \eta \) can be fixed by numerically computing \( \mathcal{V} \) at one value of \( \zeta \in (\ell_0, u_0) \).

► This makes the slow scale correction straightforwardly computable.
Figure: Safe rates $\delta_0$ and $\delta_0 + \sqrt{\varepsilon} \delta_1$ in the slow-scale stochastic volatility with negative correlation.
Figure: Boundaries $\ell_0$, $u_0$ and $\ell_0 + \sqrt{\varepsilon} \ell_1$, $u_0 + \sqrt{\varepsilon} u_1$ in the slow-scale stochastic volatility.
The slow and fast asymptotics give different quantitative approxs. to the effect of stochastic vol on portfolio choice under transaction costs, but the qualitative features are the same: lower effective safe rate, buy and sell boundaries move to the left (under negative correlation).

The constant $V_3$ in the fast case can be estimated from the implied volatility skew.

In the zero costs case, the principal Merton value function under fast SV depends on the harmonically square averaged volatility: $\sigma^2 = \langle f^{-2} \rangle$. Here, it depends on $\bar{\sigma}$. So the limit $\lambda \downarrow 0$ after the $\varepsilon$ expansion is not clear yet.

Recently Kallsen & Muhle-Karbe start from $\lambda = 0$ and SV and expand in small $\lambda$ from the no costs Merton SV problem. Comparison of interest.
The slow and fast asymptotics give different quantitative approxs. to the effect of stochastic vol on portfolio choice under transaction costs, but the qualitative features are the same: lower effective safe rate, buy and sell boundaries move to the left (under negative correlation).

The constant $V_3$ in the fast case can be estimated from the implied volatility skew.

In the zero costs case, the principal Merton value function under fast SV depends on the harmonically square averaged volatility: $\sigma_{\times}^{-2} = \langle f^{-2} \rangle$. Here, it depends on $\bar{\sigma}$. So the limit $\lambda \downarrow 0$ after the $\epsilon$ expansion is not clear yet.

Recently Kallsen & Muhle-Karbe start from $\lambda = 0$ and SV and expand in small $\lambda$ from the no costs Merton SV problem. Comparison of interest.
The slow and fast asymptotics give different quantitative approxs. to the effect of stochastic vol on portfolio choice under transaction costs, but the qualitative features are the same: lower effective safe rate, buy and sell boundaries move to the left (under negative correlation).

The constant $V_3$ in the fast case can be estimated from the implied volatility skew.

In the zero costs case, the principal Merton value function under fast SV depends on the harmonically square averaged volatility: $\sigma_x^{-2} = \langle f^{-2} \rangle$. Here, it depends on $\sigma$. So the limit $\lambda \downarrow 0$ after the $\varepsilon$ expansion is not clear yet.

Recently Kallsen & Muhle-Karbe start from $\lambda = 0$ and SV and expand in small $\lambda$ from the no costs Merton SV problem. Comparison of interest.
The slow and fast asymptotics give different quantitative approxs. to the effect of stochastic vol on portfolio choice under transaction costs, but the qualitative features are the same: lower effective safe rate, buy and sell boundaries move to the left (under negative correlation).

The constant $V_3$ in the fast case can be estimated from the implied volatility skew.

In the zero costs case, the principal Merton value function under fast SV depends on the harmonically square averaged volatility: $\sigma^{-2} = \langle f^{-2} \rangle$. Here, it depends on $\bar{\sigma}$. So the limit $\lambda \downarrow 0$ after the $\varepsilon$ expansion is not clear yet.

Recently Kallsen & Muhle-Karbe start from $\lambda = 0$ and SV and expand in small $\lambda$ from the no costs Merton SV problem. Comparison of interest.