Optimality under small transaction costs

On obtaining rigorous results

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Banff, May 16, 2014

Outline

1. The problem
2. The solution
3. The verification
4. The conclusion

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The setup

Itô process with proportional transaction costs

- Risky asset:

$$dS_t = b_t^S dt + \sigma_t^S dW_t$$

- Wealth process:

$$X_t^\varepsilon(\psi^\varepsilon, k^\varepsilon) = x + \int_0^t \psi_s^\varepsilon dS_s - \int_0^t k_s^\varepsilon ds + \Psi_t - \int_0^t \varepsilon_s d\|\psi^\varepsilon\|_s,$$

  where
  - ψ_t^ε trading strategy (= number of shares),
  - k_t^ε consumption rate,
  - x initial endowment,
  - Ψ_t random endowment stream, e.g. $\Psi_t = 0$
  - $\varepsilon_t = \varepsilon \mathcal{E}_t$ proportional transaction costs, e.g. $\mathcal{E}_t = S_t$
- ε is supposed to be "small."

Maximising expected utility

under small proportional transaction costs

- Goal:

$$\max_{(\psi^\varepsilon, k^\varepsilon)} E\left[\int_0^T u_1(t, k_t^\varepsilon)dt + u_2(X_T^\varepsilon(\psi^\varepsilon, k^\varepsilon))\right]$$

  with possibly random utility functions $u_1(t, \cdot), u_2(\cdot)$

- More precisely: determine leading-order correction to frictionless problem ($\varepsilon = 0$) for small costs ε
- in this talk: $u_2(x) = -\exp(-px)$, no consumption, $\varepsilon_t = \varepsilon S_t$

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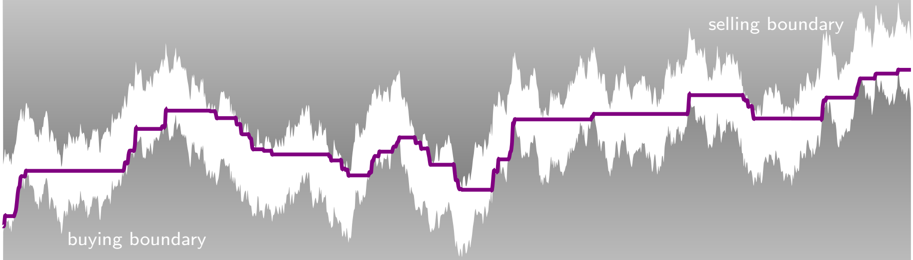
Selected and incomplete references

on expected utility optmisation under proportional transaction costs

- The structure: Magill and Constantinides (1976), Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), . . .
- Asymptotics: Shreve and Soner (1994), Janeček and Shreve (2004), Soner and Touzi (2013), K and Muhle-Karbe (2013), . . .
- Utility indifference pricing and hedging: Hodges and Neuberger (1989), Davis, Panas, Zariphopoulou (1993), . . .
- Utility imdifference and asymptotics: Whalley and Wilmott (1997), Barles and Soner (1998), Bichuch (2011), . . .
- Shadow prices: Jouini and Kallal (1995), Cvitanic and Karatzas (1996), K and Muhle-Karbe (2010), . . .

width of no-trade corridor: $\Delta \mathrm{NT}_t = \sqrt[3]{12 R_t \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon_t}$



certainty equivalent loss: $\Delta \mathrm{CE} = E^Q \left[\int_0^T \frac{(\Delta \mathrm{NT}_t)^2}{8 R_t} d\langle S \rangle_t \right]$

Leading-order asymptotics

- Asymptotic no-trade region: $[\overline{\mathrm{NT}}_t - \Delta \mathrm{NT}_t, \overline{\mathrm{NT}}_t + \Delta \mathrm{NT}_t]$,

$$\boxed{\Delta NT_t = \sqrt[3]{\frac{3R_t}{2}\frac{d\langle\varphi\rangle_t}{d\langle S\rangle_t}\varepsilon_t}}$$

  where here (for exponential utility)
  - $\overline{\mathrm{NT}} = 0$,
  - indirect risk tolerance $R = 1/p$.
- Asymptotic certainty equivalent of utility loss:

$$\boxed{E_Q\left[\int_0^T \frac{(\Delta \mathrm{NT}_t)^2}{2R_t} d\langle S\rangle_t\right]}$$

  with dual minimiser Q of frictionless problem (here: MEMM)

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Some references

on verifying first-order approximations

- Based on dynamic programming:
  - Careful analysis and expansion of the value function:
    Shreve and Soner (1994), Janeček and Shreve (2004, 2010), Bichuch and Shreve (2013), Bichuch (2011, 2012), ...
  - Homogenisation approach:
    Soner and Touzi (2013), Possamai, Soner, Touzi (2012), Altarovici, Muhle-Karbe, Soner (2013) Bouchard, Moreau, Soner (2013), Moreau, Muhle-Karbe, Soner (2014), Possamai and Royer (2014), ...
- Based on shadow prices:
  - Expanding the exact solution:
    Gerhold, Muhle-Karbe, Schachermayer (2012), Gerhold, Guasoni, Muhle-Karbe, Schachermayer (2014), Guasoni and Muhle-Karbe (2013), ...
  - Here (joint work with Shen Li):
    find explicit nearly optimal candidates for primal and dual problem

A rigorous theorem

Idea of the proof (cf. Henderson 2002, Kramkov & Sirbu 2006)

1. Existence of the process $\Delta\varphi$ as solution to a Skorohod problem with reflection.
2. Definition of $\Delta\tilde{S}$ as a function of $\Delta\varphi$.
3. Leading-order approximation of the corresponding expected utility by Taylor expansion.
4. Consider MMM $\tilde{Z}^{\varepsilon}$ for $\tilde{S}^{\varepsilon}$ relative to Q.
5. Compute leading-order approximation of Lagrange dual function at $\tilde{Z}^{\varepsilon}$. Observe that lower and upper bound coincide to leading order.
6. Replace $\Delta\varphi$ in expression for certainty equivalent loss by ΔNT, using the ergodic property of reflected Brownian motion.

Draw a picture!

Regularity conditions

in our previous version

- There exists EMM for S with finite relative extropy.
  $\rightsquigarrow$ MEMM Q and frictionless optimiser φ exist.
- φ and $\rho := \frac{d\langle\varphi\rangle}{d\langle S\rangle} > 0$ are Itô processes.
- Set $Y := \log(S)$, $\pi := \varphi S$, $\eta := \rho S^4$.

$$E_Q\left[\sup_{t\in[0,T]} |g|_t^n\right] < \infty, \quad n \in \mathbb{N}$$

  for processes $g \in \{\pi, \eta, 1/\eta, b^\pi, b^\eta, c^Y, 1/c^Y, c^\pi, c^\eta\}$.
- $E_Q\left[\exp(|9p\int_0^T \varphi_t dS_t|)\right] < \infty$
- c^π, c^Y, c^η are continuous.

Regularity conditions in the literature

which fail if $d\langle\varphi\rangle_t/dt = 0$

Bichuch (2011):

Assumption 3.1 *We will assume that the contingent claim payoff function* $g : [0,\infty) \to [0,\infty)$ *is four times continuously differentiable, and such that* $g(S) - g'(S)S$, $S^2 g''(S)$, $S^3 g^{(3)}(S)$ *and* $S^4 g^{(4)}(S)$ *are all bounded for* $S \in [0,\infty)$.

Assumption 3.2 *We will also assume that* $\frac{e^{-rT}(\mu-r)}{\gamma\sigma^2} - \sup_{S\in[0,\infty)} S^2 |g''(S)| \geq \varepsilon_1$, *for some* $\varepsilon_1 > 0$. *The supremum above is finite by Assumption 3.1*

Soner and Touzi (2013):

equation is nondegenerate away from the origin. For later use we record that there exist two constants $0 < \alpha_* \leq \alpha^*$ so that

$$0 < \alpha_* \leq \frac{\alpha(s,z)}{z} \leq \alpha^* \qquad \forall\ s, z \in \mathbb{R}_+. \tag{5.3}$$

We will not attempt to verify the above hypothesis. However, in the power utility

A rigorous theorem

Statement

- Define ΔNT as before.
- There exists $\varphi^\varepsilon = \varphi + \Delta\varphi = \varphi^{\varepsilon\uparrow} - \varphi^{\varepsilon\downarrow}$ with values in $[\varphi - \Delta NT, \varphi - \Delta NT]$, where $\varphi^{\varepsilon\uparrow}, \varphi^{\varepsilon\downarrow}$ increase only at the boundary.
- Set

$$\tau^\varepsilon := \inf\left\{t \in [0, T] : |X_t(\varphi^\varepsilon) - (x + \varphi \bullet S_t)| > 1 \text{ or } |X_t(\varphi^\varepsilon)| > \varepsilon^{-4/3}\right\}$$

- Then $\varphi^\varepsilon 1_{[\![0,\tau^\varepsilon]\!]}$ is optimal to the leading order with certainty equivalent loss as stated earlier.

Draw another picture!

Relaxed regularity conditions

by trading more carefully

- There exists EMM for S with finite relative extropy.
  $\rightsquigarrow$ MEMM Q and frictionless optimiser φ exist.
- φ and $\rho := \frac{d\langle\varphi\rangle}{d\langle S\rangle}$ are Itô processes.
- $$\begin{aligned} E_Q\left[\int \left((1+S_t^2)c_t^\varphi\right)^{1\frac{5}{12}} dt\right] &< \infty, \\ E_Q\left[\int \left((1+|S_t|)b_t^\varphi\right)^{1\frac{5}{12}} dt\right] &< \infty, \\ E_Q\left[\int \left(c_t^S\right)^3 dt\right] &< \infty, \\ E_Q\left[\sup_{t\in[0,T]} (1+|\varphi_t|)|S_t|\right] &< \infty \end{aligned}$$
- $E_Q\left[\exp(|9p\int_0^T \varphi_t dS_t|)\right] < \infty$
- $c^\varphi, c^S, c^\varrho, 1/c^S, b^\varphi, b^\varrho$ are continuous.

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Main conclusions

concerning leading order asymptotics

- Verfication is sometimes painful.
- We get ahead slowly but steadily.
  (Seem to be able to relax Bichuch's condition to make Whalley and Willmott hold for calls/puts for low stock drift.)
- Please don't kill me if I made a mistake.