A convergence result for the Emery topology and a variant of the proof of the Fundamental Theorem of Asset Pricing

Christa Cuchiero
(based on joint work with Josef Teichmann)

Technical University Vienna

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- The motivation for this question stems from the proof of the Fundamental Theorem of Asset Pricing (FTAP) in continuous time where one crucial part consists in proving exactly this statement under (NFLVR) and a certain maximality assumption.
- Is it possible to transform this to a criterion/theorem interesting in its own right?

Goal of this talk

- Present a general principle for sequences of semimartingales, the so-called P-UT property, which allows together with a maximality assumption to conclude the desired convergence in the Emery topology for up-convergent sequences of semimartingales.

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- Present a general principle for sequences of semimartingales, the so-called P-UT property, which allows together with a maximality assumption to conclude the desired convergence in the Emery topology for up-convergent sequences of semimartingales.
- Application to the setting of FTAP, which allows to obtain a variant of one part of the proof by showing that the P-UT property is implied by (NFLVR).

Setting and notation

- We consider a finite time horizon $T = 1$ and a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$. For a process Z we write $|Z|_1^* = \sup_{t\in[0,1]} |Z_t|$.

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- The set of semimartingales starting at 0 on the time interval $[0, 1]$ is denoted by $\mathbb{S}$.

- We equip $\mathbb{S}$ with the Emery metric

$$\sup_{H\in b\mathcal{E}, \|H\|_\infty \leq 1} E[|(H \bullet (X - Y))|_1^* \wedge 1] = d_E(X, Y),$$

  making it a complete metric space. Here, $b\mathcal{E}$ denotes the set of bounded simple predictable strategies.

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- Pathwise uniform convergence in probability is metrized by

$$E[|X - Y|_1^* \wedge 1] = d(X, Y),$$

which makes the set of càdlàg processes a complete metric space.

Definition of the (P-UT) property

Let us introduce the (P-UT) property (cf. Jacod and Shiryaev (2003)), originally introduced by Stricker (1985) under Condition (*) and Jakubowski, Mémin and Pagès (1989) under the name (UT) (for "uniforme tension").

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Definition

We say that a sequence $(X^n)_{n\geq 0}$ of semimartingales satisfies the (P-UT) property (predictably uniformly tight) if for every $t > 0$ the family of random variables $\{(H \bullet X^n)_t : H \in b\mathcal{E}, \|H\|_\infty \leq 1, n \in \mathbb{N}\}$ is bounded in L^0, that is,

$$\lim_{c\uparrow\infty} \sup_{H,n\in\mathbb{N}} P[|(H \bullet X^n)|_t > c] = 0\,.$$

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$$\lim_{c \uparrow \infty} \sup_{H, n \in \mathbb{N}} P[|(H \bullet X^n)|_t > c] = 0 .$$

- The (P-UT) property can be seen as boundedness in the Emery topology.
- Interpretation from mathematical finance: No unbounded profit or loss with simple, predictable, bounded by 1, long or short positions

Convergence of stochastic integrals under the (P-UT) property

"...the following theorem is in fact the very reason for introducing the (P-UT) property" (Jacod, Shirayev (2003))

Theorem (Jakubowski, Mémin and Pagès (1989))

Let $(H^n)_{n\geq 0}$ be a sequence of adapted càdlàg processes and $(X^n)_{n\geq 0}$ a sequence of semimartingales satisfying the (P-UT) property. If (H^n, X^n) converges pathwise uniformly in probability to (H, X), then $(H^n_- \bullet X^n)$ converges to $(H_- \bullet X)$ pathwise uniformly in probability as well.

Convergence of stochastic integrals and the quadratic variation under the (P-UT) property

Corollary (Jakubowski, Mémin and Pagès (1989))

Let $(X^n)_{n\geq 0}$ be a sequence of semimartingales satisfying the (P-UT) property. If (X^n) converges pathwise uniformly in probability to X, then $[X^n, X^n]$ converges to $[X, X]$ pathwise uniformly in probability as well.

Sketch of the proof

This follows from the above theorem by Itô's integration by parts formula.

Emery convergence for the local martingale and the big jump part under (P-UT) and up-convergence

For a sequence of semimartingales (X^n) with $X_0^n = 0$ and some $C > 0$ we consider the following decomposition

$$X^n = B^{n,C} + M^{n,C} + \check{X}^{n,C},$$

where $\check{X}_t^{n,C} = \sum_{s \leq t} \Delta X_s^n 1_{\{|\Delta X_s| > C\}}$, $B^{n,C}$ is the predictable, finite variation part and $M^{n,C}$ the local martingale part of the canonical decomposition of the special semimartingale $X^n - \check{X}^{n,C}$.

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Theorem (Mémin and Slominski (1991))

Let (X^n) be a sequence of semimartingales with $X_0^n = 0$, which converges pathwise uniformly in probability to X and satisfies the (P-UT) property. Then there exists some $C > 0$ such that $M^{n,C} \to M^C$ and $\check{X}^{n,C} \to \check{X}^C$ in the Emery topology and $B^{n,C} \to B^C$ pathwise uniformly in probability.

Remarks on the proof

Preliminaries on Emery convergence

- A sequence of adapted finite variation processes (X^n) converges in the Emery topology to 0 if $\lim_{n\to\infty} TV(X^n)_1 \to 0$ in probability.
- A sequence of local martingale (X^n) with $|\Delta X^n|_1^* \leq C$ (uniformly in n) converges in the Emery topology to 0 if $\lim_{n\to\infty}[X^n, X^n]_1 \to 0$ in probability.

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Sketch of the proof (of the Mémin and Slominski Theorem)

The process $\check{X}^{n,C} - \check{X}^C$ converges in variation and thus in the Emery topology to 0. Moreover, (P-UT) and up convergence imply

$$[(X^n - \check{X}^{n,C}) - (X - \check{X}^C), (X^n - \check{X}^{n,C}) - (X - \check{X}^C)]_1 \to 0$$

in probability. From this $[M^{C,n} - M^C, M^{C,n} - M^C]_1 \to 0$ in probability and thus $M^{n,C} - M^C \to 0$ in the Emery topology can be deduced. Pathwise uniform convergence in probability of $B^{n,C} \to B^C$ is obtained by difference.

How to deal with the finite variation part (without big jumps)

- For a sequence of semimartingales (X^n), consider for every $r \in \mathbb{N}$ the following sets

$$\mathcal{Y}^r = \{H \bullet X^n + G \bullet X^m \,|\, n, m \geq r, H, G \text{ bounded predictable } \|H\|_\infty \leq 1, \|G\|_\infty \leq 1, HG = 0\},$$

and denote by $\mathcal{Z}^r = \operatorname{conv}(\mathcal{Y}^r)$ its convex hull. Moreover, we denote $\mathcal{V}_1^r = \{Z_1 \,|\, Z \in \mathcal{Z}^r\}$ and by $\widehat{\mathcal{V}}_1^r$ its closure in L^0. Define $\widehat{\mathcal{V}}_1 = \bigcap_{r \geq 0} \widehat{\mathcal{V}}_1^r$.

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Lemma

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Lemma

Let (X^n) be a sequence of semimartingales satisfying the (P-UT) property. Then the set $\widehat{\mathcal{V}}_1$ is bounded in L^0.

- From this it follows that $\widehat{\mathcal{V}}_1$ contains a maximal element.

A convergence result in the Emery topology

Theorem

Let (X^n) be a sequence of semimartingales satisfying the (P-UT) property. Take a sequence $(Z^n) \in \mathcal{Z}^1$ such that (Z_1^n) converges a.s. to a maximal element Z_1 in $\widehat{\mathcal{V}}_1$. Then (Z^n) converges in the Emery topology.

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Proof.

- Due to the P-UT property of (X^n), $\mathcal{Z}^1$ is bounded in L^0 and the sequence (Z^n) satisfies the (P-UT) property.
- Moreover, as in the proof of FTAP by Delbaen and Schachermayer the maximality of the end value allows to conclude up-convergence.
- An application of Mémin and Slominski's theorem yields Emery convergence of the local martingale and the big jump part of Z^n in the Emery topology.
- Using this together with the maximality of Z_1, an adaptation of the last part in the proof of FTAP yields convergence of the finite variation part in the Emery topology.

□

Towards the setting of FTAP

- The conclusion of the above result holds for sequences of portfolio wealth processes bounded from below by -1 under the (NUPBR) (No unbounded profit with bounded risk) condition, which means L^0-boundedness of portfolio end values.

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- The conclusion of the above result holds for sequences of portfolio wealth processes bounded from below by -1 under the (NUPBR) (No unbounded profit with bounded risk) condition, which means L^0-boundedness of portfolio end values.
- The crucial issue consists in proving that (NUPBR) implies (P-UT).

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- Their beautiful and impressive proof builds on deep insights and is in some parts quite tricky. No essential simplification has been obtained since then.
- It was however realized soon that the presented proof can be transformed to a more abstract setting of portfolio wealth processes (Y. Kabanov (1996)).

Definition of portfolio wealth processes

Definition

- We consider a convex set $\mathcal{X}_1 \subset \mathbb{S}$ of semimartingales
  - starting at 0,
  - bounded from below by -1,
  - being closed in the Emery topology, and
  - satisfying the following concatenation condition: for all $X, Y \in \mathcal{X}_1$ and all bounded predictable strategies $H, G \geq 0$, with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}_1$.

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- We denote by $\mathcal{X}$ the set $\mathcal{X} = \cup_{\lambda > 0} \lambda \mathcal{X}_1$ and call its elements admissible portfolio wealth processes.

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- We denote by K_0, respectively K_0^1 the evaluations of elements of $\mathcal{X}$, respectively $\mathcal{X}_1$, at final time $T = 1$.

Remarks

- The setting of portfolio wealth processes is an abstract formulation of the classical setting of Delbaen and Schachermayer, where trading (in an admissible way) with respect to a d-dimensional semimartingale S is considered.
- In particular, the above requirements (admissibility, concatenation, closedness in the Emery topology) are those which are satisfied in the case of $\mathcal{X}_1 = \{H \bullet S \,|\, H \text{ 1-admissible } \}$. Closedness in the Emery topology of this set was proved by Mémin (1980).

Notions of No Arbitrage

(NA) The set $\mathcal{X}$ is said to satisfy No Arbitrage if $K_0 \cap L^0_{\geq 0} = \{0\}$ which can be shown to be equivalent to $C \cap L^\infty_{\geq 0} = \{0\}$, with $C = (K_0 - L^0_{\geq 0}) \cap L^\infty$.

(NFLVR) The set $\mathcal{X}$ is said to satisfy No free lunch with vanishing risk if

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}$ denotes the norm closure in L^∞.

(NFL) The set $\mathcal{X}$ is said to satisfy No free lunch if

$$\overline{C}^* \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}^*$ denotes the weak-$*$-closure in L^∞.

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Definition

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $\mathbb{E}_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

(NFLVR) $\Leftrightarrow$ (ESM)

- It is a consequence of Hahn-Banach's Theorem (the Kreps-Yan Theorem) that (NFL) implies the existence of an equivalent measure $Q \sim P$ such that $E_Q[f] \leq 0$ for all $f \in C$ and hence for all $f \in K_0$.

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- We obviously have $(NFL) \Rightarrow (NFLVR) \Rightarrow (NA)$, but it is a deep insight that under (NFLVR) it holds that $C = \overline{C}^*$, i.e. the cone C is already weak-$*$-closed and (NFL) holds.

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- We obviously have $(NFL) \Rightarrow (NFLVR) \Rightarrow (NA)$, but it is a deep insight that under (NFLVR) it holds that $C = \overline{C}^*$, i.e. the cone C is already weak-$*$-closed and (NFL) holds.
- The goal is to show (NFLVR) $\Rightarrow C = \overline{C}^*$, which proves the Fundamental Theorem of Asset Pricing:

Theorem (Delbaen and Schachermayer (1994))

Under (NFLVR) the cone C is weak $*$-closed, hence (NFL) holds, which is equivalent to (ESM). In other words: (NVLVR) $\Leftrightarrow$ (ESM).

First part of the original proof by Delbaen and Schachermayer

- The convex cone $C = (K_0 - L^0_{\geq 0}) \cap L^\infty$ is closed with respect to the weak-$*$-topology if and only if $C_0 = K_0 - L^0_{\geq 0}$ is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f it holds that $f \in C_0$.

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- Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $g_n \in K_0$, i.e. $g_n = Y^n_1$ with $Y^n \in \mathcal{X}$ such that $f_n \leq g_n = Y^n_1$.

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- By (NA) it follows that $Y^n \in \mathcal{X}_1$.

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- Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $g_n \in K_0$, i.e. $g_n = Y^n_1$ with $Y^n \in \mathcal{X}$ such that $f_n \leq g_n = Y^n_1$.
- By (NA) it follows that $Y^n \in \mathcal{X}_1$.
- By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \operatorname{conv}(Y^n, Y^{n+1}, \ldots)$ such that $\widetilde{Y^n_1} \to \widetilde{h}_0 \geq f$ almost surely for some finite $\widetilde{h}_0$. This implies that the set $\widehat{K}^1_0 \cap \{g \in L^0 \,|\, g \geq f\}$, where $\widehat{K}^1_0$ denotes the closure of K^1_0 in L^0 is non-empty. Since it is also bounded and closed, a maximal element h_0 exists.

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- Since $h_0 \in \widehat{K}_0^1$, we can find a sequence of semimartingales $X^n \in \mathcal{X}_1$ such that $X_1^n \to h_0$ almost surely.

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- This "maximal" sequence of semimartingales $X^n \in \mathcal{X}_1$ converges pathwise uniformly in probability, i.e. $|X^n - X|_1^* \to 0$ in probability for some càdlàg process X.
- It is now the goal to show that indeed $X^n \to X$ in the Emery topology, an apparently much stronger statement. If we can show this, it follows that $h_0 = \lim X_1^n = X_1 \in K_0^1$ since $X \in \mathcal{X}_1$ by the closedness of $\mathcal{X}_1$ in the Emery topology. This then implies that $f \in C_0$ and we are done.

Comments on the original proof

- By performing an equivalent measure change to deal with special L^2-semimartingales, the original proof proceeds by showing Emery convergence of the local martingale part and the finite variation part of forward convex combinations of (X^n). This is achieved by a series of quite tricky lemmas.

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Goal: Show the Emery convergence of the “maximal” up-convergent sequence of semimartingales (X^n) directly via the (P-UT) property

Convergence in the Emery topology

The series of lemmas can be summarized by the following theorem.

Theorem

Let $\mathcal{X}_1$ satisfy (NUPBR) and let $(X^n) \in \mathcal{X}_1$ be a sequence of semimartingales, whose end values (X_1^n) converges almost surely to a maximal element in $\widehat{K_0^1}$. Then $X^n \to X$ in the Emery topology.

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Proof.

The maximality of the end value allows to conclude up-convergence. The crucial part then consists in proving ((NUPBR) $\Rightarrow$ (P-UT)). The remaining part follows as before from Memin and Slominski's theorem and a modification of the last part of the proof of FTAP, namely Emery convergence of the finite variation part. □

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Remark

Due to the concatenation property (NUPBR) allows to deduce directly that $\{(H \bullet X)_1 \,|\, X \in \mathcal{X}_1, H \geq 0$ bounded predictable s.t. $H \bullet X \geq -1\}$ is bounded in L^0, but in order to get it for all $H \in b\mathcal{E}$ with $\|H\|_\infty \leq 1$ more work has to be done.

Supermartingale deflator

Definition

A positive càdlàg adapted process D is called supermartingale deflator for $1 + \mathcal{X}_1$ if D is strictly positive, $D_0 \leq 1$ and $D(1 + X)$ is a supermartigale for all $X \in \mathcal{X}_1$.

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Theorem (Karatzas and Kardaras (2007)/ Kardaras (2013))

Assume (NUPBR) for $\mathcal{X}_1$, then there exists a supermartingale deflator D.

(P-UT) property for supermartingales

Lemma

Let (Z^n) be a sequence of non-negative supermartingales such that $Z_0^n \leq K$ for all $n \in \mathbb{N}$ and some $K > 0$. Then (Z^n) satisfies the P-UT property.

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Proof.

By an inequality of Burkholder for non-negative supermartingales S and processes $H \in b\mathcal{E}$ with $\|H\| \leq 1$ it holds that

$$cP[|(H \bullet S)|_1^* \geq c] \leq 9\mathbb{E}[|S_0|]$$

for all $c \geq 0$. Applying this inequality to Z^n and letting $c \to \infty$ yields the P-UT property. □

(P-UT) property for sequences in $\mathcal{X}_1$

Proposition

Let $\mathcal{X}_1$ satisfy (NUPBR) and let (X^n) be any sequence of semimartingales in $\mathcal{X}_1$. Then (X^n) satisfies the P-UT property.

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Proof.

The (P-UT) property of the supermartingales $(Z^n) := (D(1 + X^n))$ can be easily transferred to the sequence (X^n). It relies on Itô's integration by parts formula and the fact that $(H^n_- \bullet S^n)$ satisfies (P-UT), if (S^n) is a sequence of semimartingales satisfying (P-UT) and (H^n) a sequence of adapted càdlàg processes such that $(|H^n|_1^*)_n$ is bounded in L^0. □

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Proposition

Assume (NUPBR). Let $(X^n)_{n\geq 0}$ together with an adapted, càdlàg process X such that $|X^n - X|_1^* \to 0$ in probability as $n \to \infty$ be a sequence in $\mathcal{X}_1$.

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- *Then for every $C > 0$ there exists a decomposition $X^n = M^n + B^n + \check{X}^n$ into a local martingale M^n, a predictable, finite variation process B^n and the finite variation process of big jumps $\check{X}_t^n = \sum_{s\leq t} \Delta X_s^n 1_{\{|\Delta X_s| > C\}}$, for $n \geq 0$, such that jumps of M^n and B^n are bounded by $2C$ uniformly in n.*

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- *Then the sequence $(\mathrm{TV}(\check{X}_1^n))_{n\geq 0}$ of total variations of $\check{X}^n$ is bounded in L^0 and $(\check{X}^n)_{n\geq 0}$ satisfies the P-UT property.*

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Proposition

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- *The sequence $(X^n)_{n\geq 0}$ satisfies P-UT.*

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- ...from one key lemma proved by Delbaen and Schachermayer with an additional analysis of the finite variation part.
- The P-UT property corresponds to boundedness in the Emery topology with an interpretation from mathematical finance. It is therefore natural to investigate this property.

Conclusions

- The (P-UT) property allows to conclude Emery convergence of sequences of semimartingales whose end values satisfy a certain maximality assumption.
- This allows to obtain a variant of the the second part of the proof of FTAP without performing a measure change and passing to forward convex combinations.

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- The Emery convergence result for sequences in $\mathcal{X}_1$ can be summarized by

$$\text{(P-UT)} + \text{maximal end value} \Rightarrow \text{Convergence in Emery}$$

- Thank you for your attention!