The Schoenflies Conjecture and its spinoffs

Banff

25 March 2014

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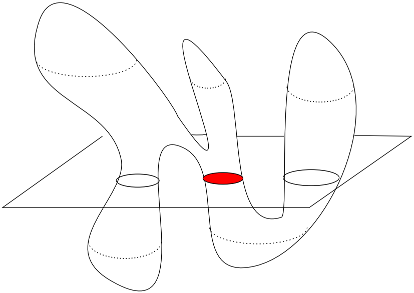

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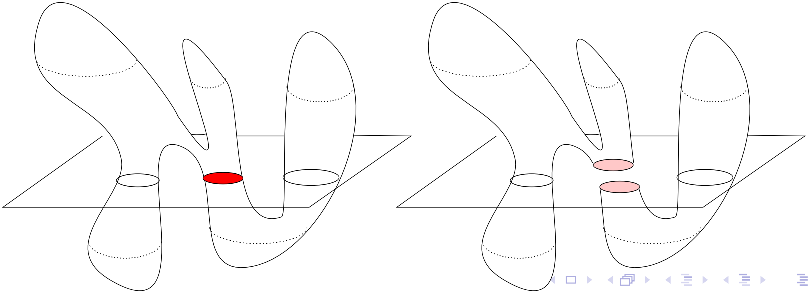

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- Bad news: $P \cap S^3_t$ is a surface in S^3. These are much more complicated than circles in the plane.
- Good news: Surfaces morphing in S^3 are fun to think about!

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  (Since X^{capped} is a homotopy 4-sphere.)

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For M^m imbedded in $N^m \times I$, can isotope M so that, for some handle structure:

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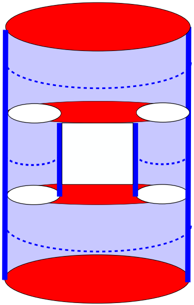
- *The handles appear in increasing order of index*



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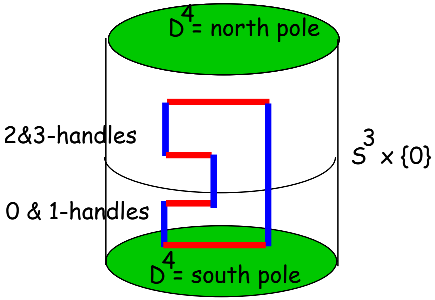

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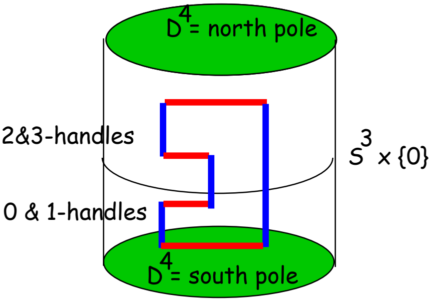

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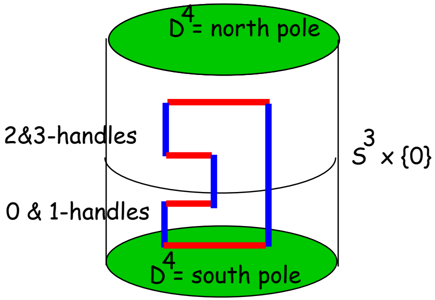

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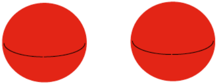

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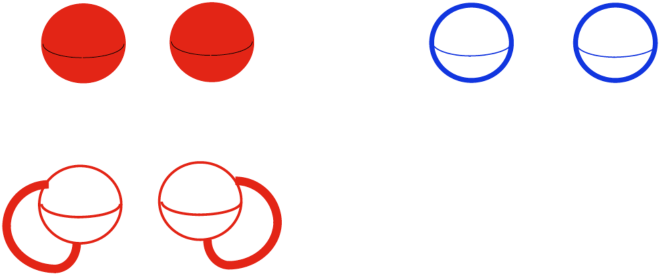

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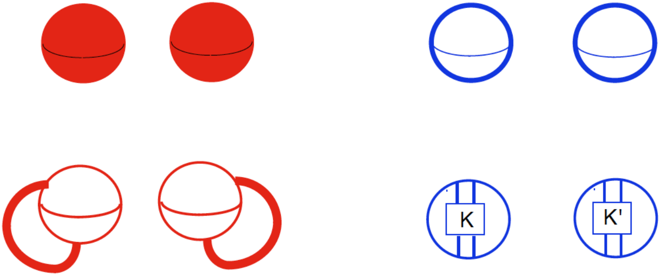

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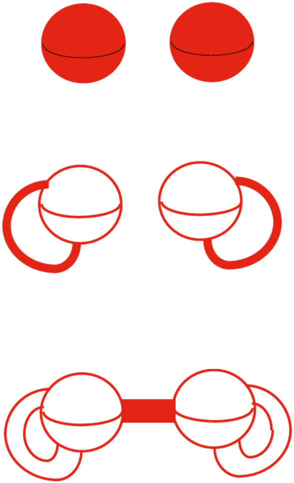

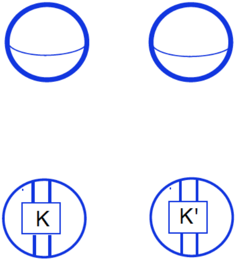

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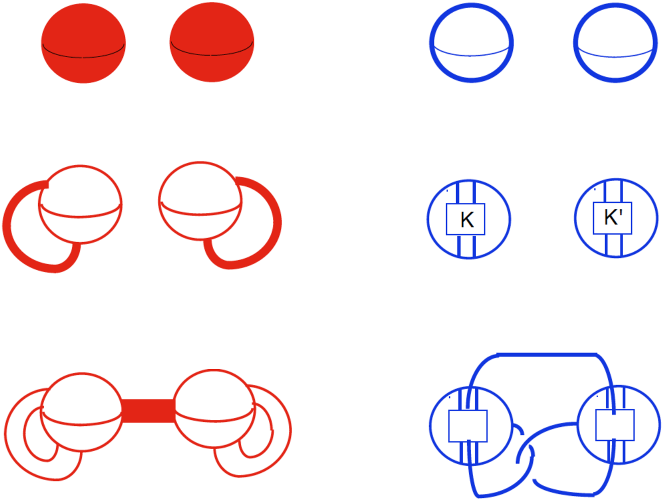

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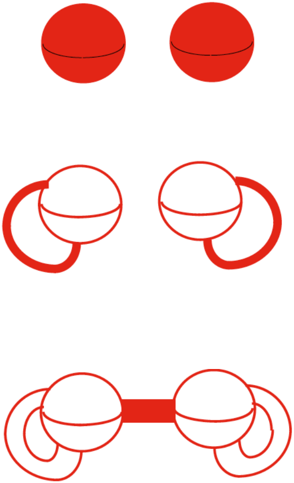

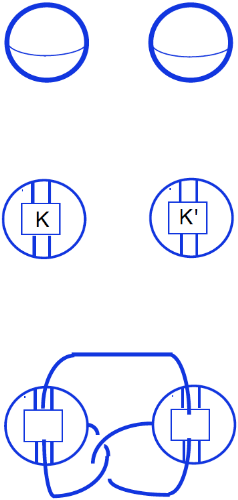

Then do the reverse!

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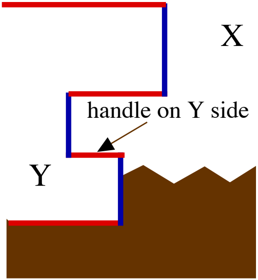
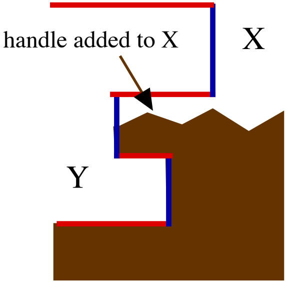
And symmetrically.





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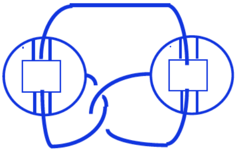

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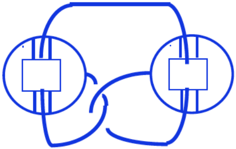
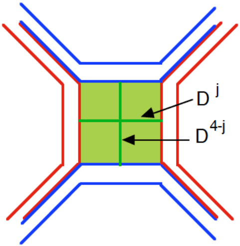


When reverse: outside 1-handle dual to inside 2-handle...
$\implies X \cong S^1 \times D^3 \natural S^1 \times D^3$

There are then two types of duality on X and Y:
Consider a j-handle added to X:

- As a j-handle in a 4-dimensional manifold X it is dual to a $(4 - j)$-handle in X



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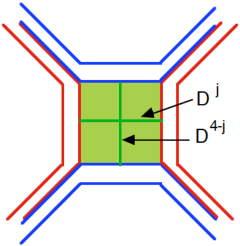
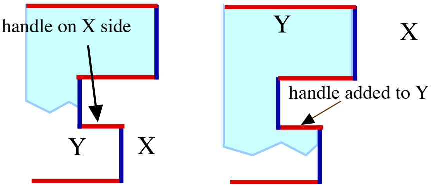
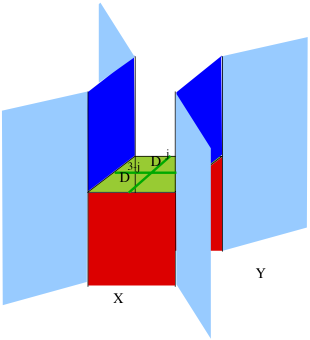




- Since it's represented by j-handle on Y-side as water rises $\Longrightarrow$ $(3-j)$-handle lying on X side as 'hydrogen descends'.



D
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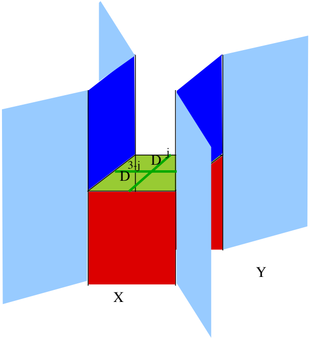
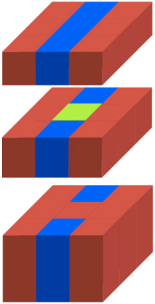
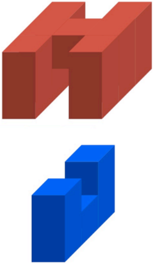
D
j
X
Y







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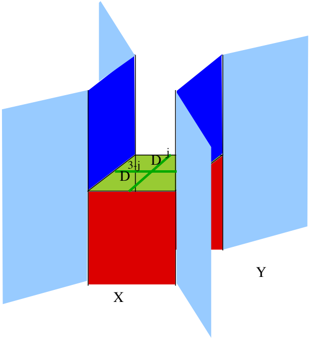

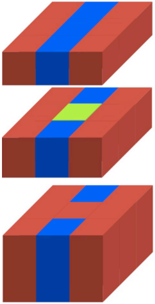

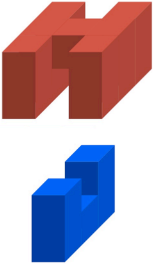

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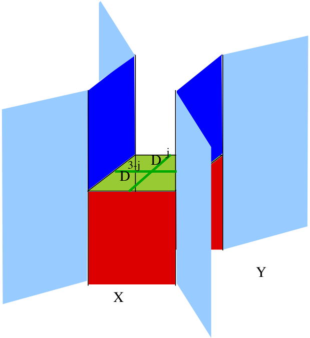

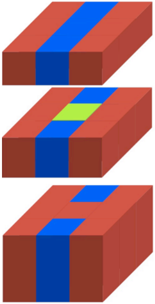

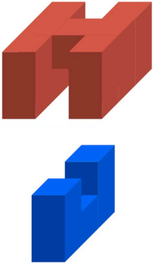

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Can show from dualities that Y is homotopy ball with only 0- and 1-handles $\implies Y \cong D^4 \implies X \cong D^4$. □

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- Is focusing on genus a good idea? Eaton (1980's), Agol-Freedman (2013) think not.

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- This suggests a focus on special case of Heegaard embeddings.

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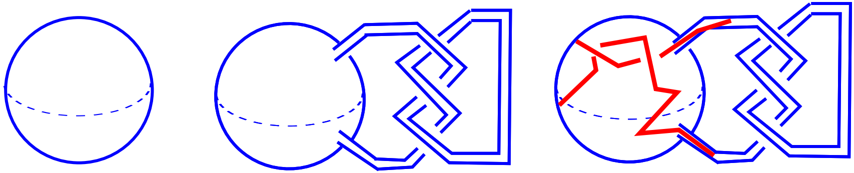

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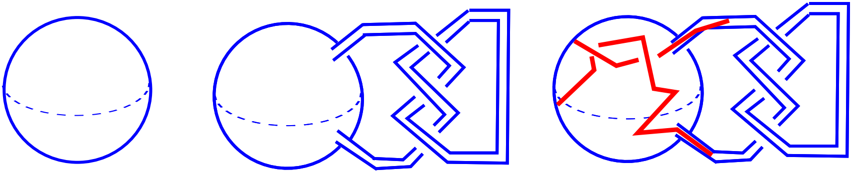

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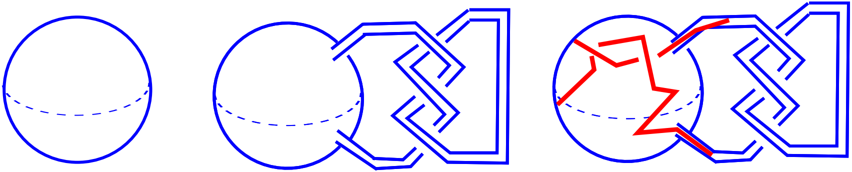

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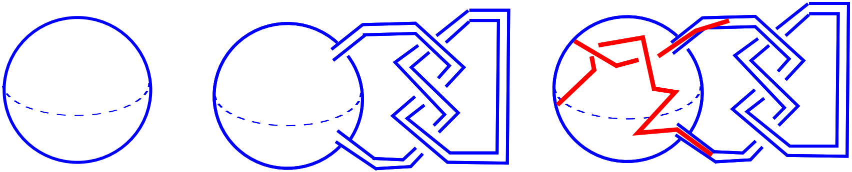

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Contradicts $P \cong S^3$.

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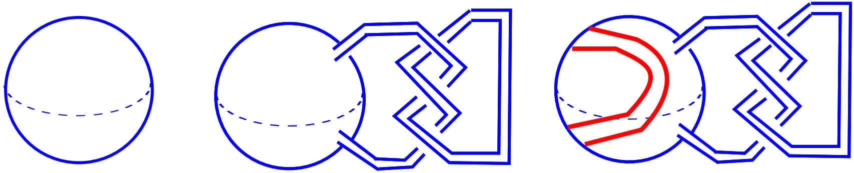

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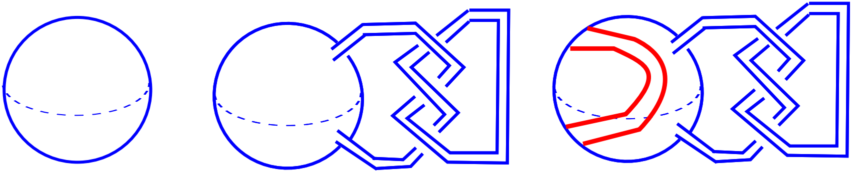

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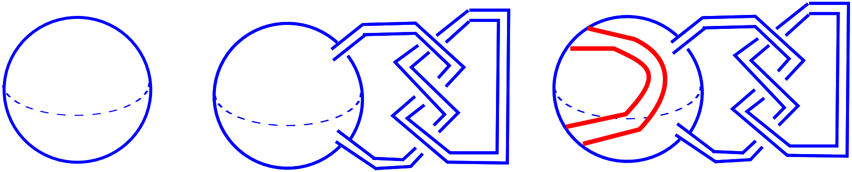

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Regardless, know X will have only one 1-handle and one 2-handle (as will Y).

Lucky break - can invoke Gabai's celebrated Property R:

Theorem (Property R Gabai 1987)

0-framed surgery on knot $K \subset S^3$ gives $S^1 \times S^2 \implies K$ is unknot.

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More general point: big 3-manifold theorem is useful for this 4-dimensional question.

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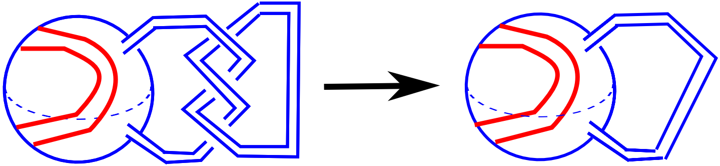

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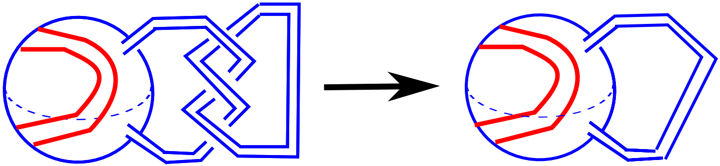

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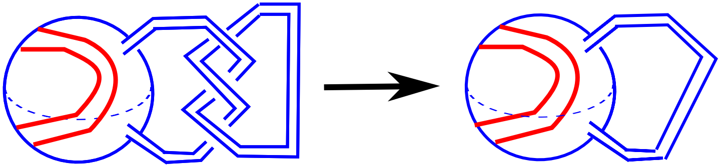

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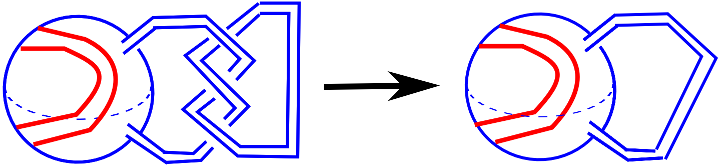

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- If can show $X' \cong D^4$ then: $\implies Y \cong D^4 \implies X \cong D^4$.

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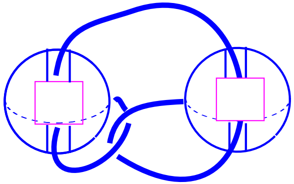

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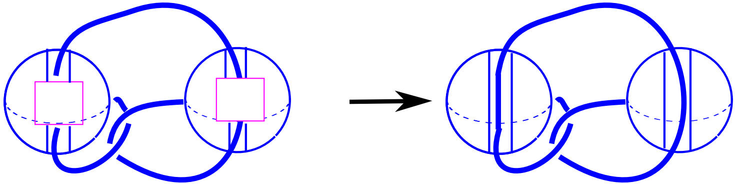

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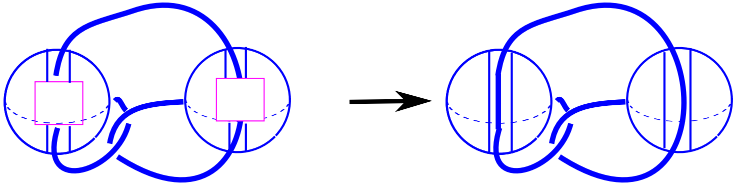

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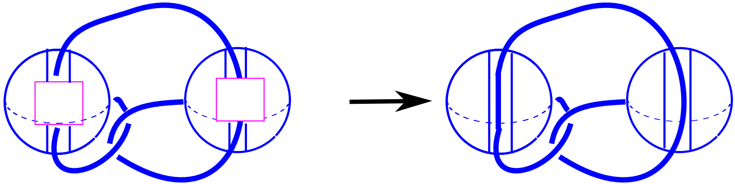

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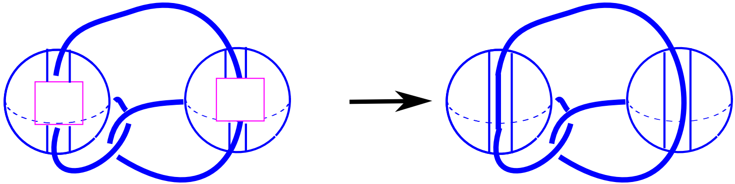

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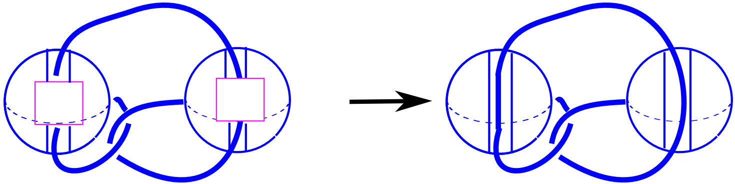

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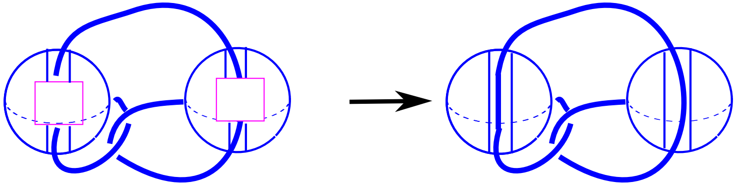

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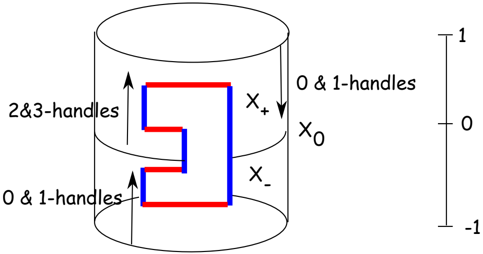
- If doesn't compress then get knotted companion tori, which describe reimbedding.

Let

- $X_0 = X \cap S^3_0$
- $X_- = X \cap S^3 \times [-1, 0]$
- $X_+ = X \cap S^3 \times [0, 1]$



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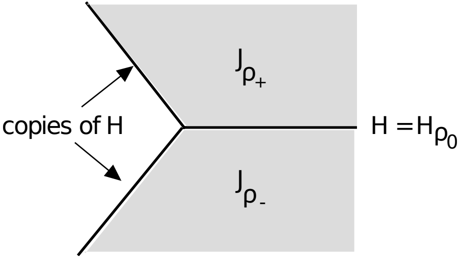

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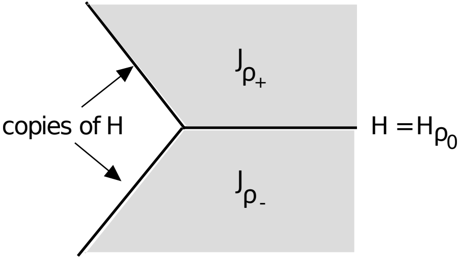

Looks like $\frac{2}{3}$ of a Gay-Kirby trisection!

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(Since $\rho_- + \rho_+ = 3, \rho_- \leq \rho_+ \implies \rho_- = 1$.)

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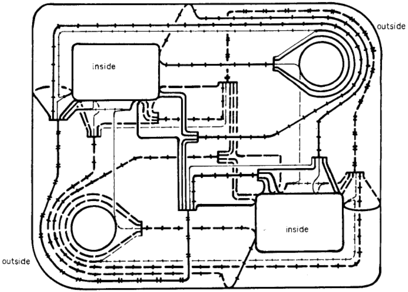

Theorem (Gompf 1991)

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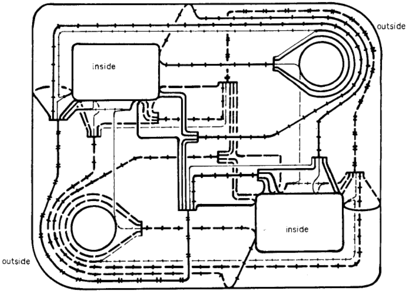

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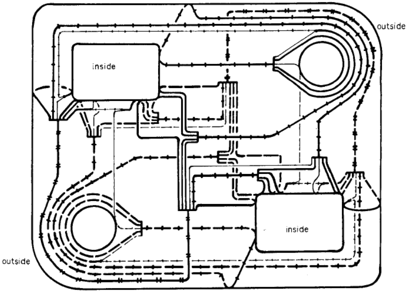

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- *$X \cong B^4$, but proof requires stabilizing (i.e. increased genus).*

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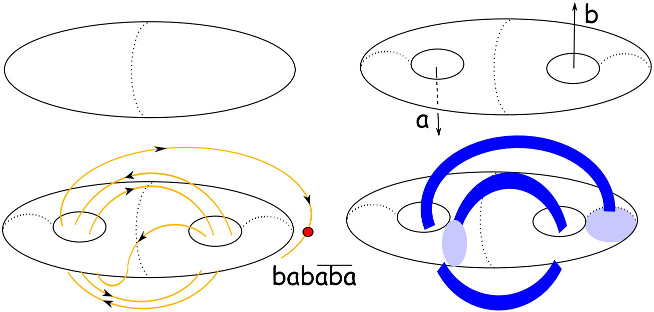
(Near) example: The first three 1-handles.



Can the picture be simplified?

It already has been! - from Akbulut-Kirby example.

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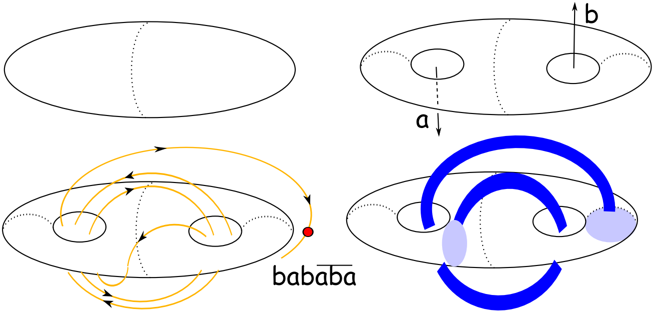
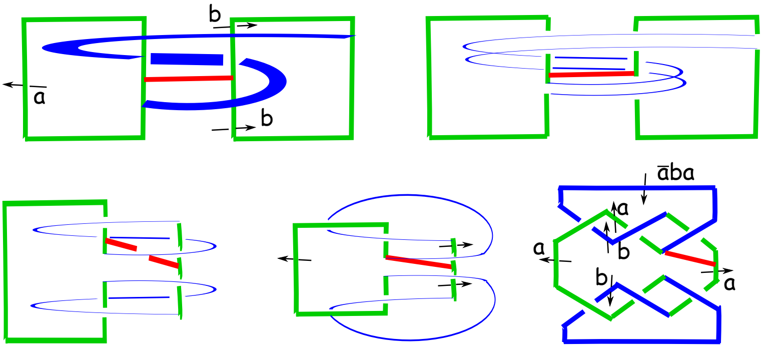


X_- unaffected by last 1-handle $\Longrightarrow$ $\pi_1(X_-) \cong \langle a, b \mid \rangle = \mathbb{Z} * \mathbb{Z}$.

The first 2-handle:



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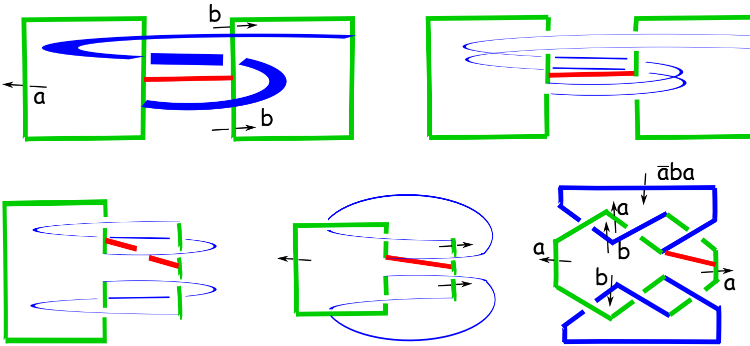

Now $\pi_1(X_-) \cong < a, b \mid aba = bab >= \pi_1(S^3 - \textit{trefoil knot})$.

The second 2-handle? If in Y then it adds relator to $\pi_1(X_-)$

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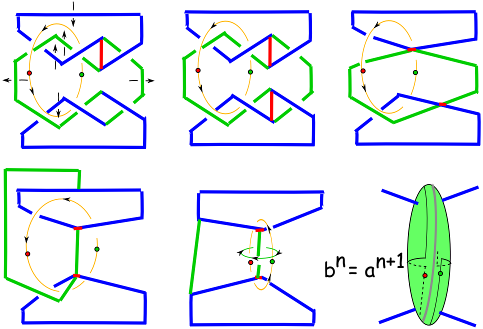

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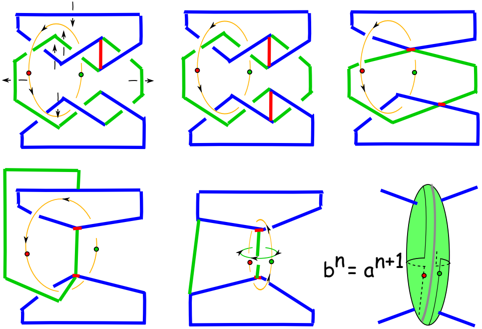

Computation shows: $\pi_1(X) \cong < a, b \mid aba = bab, a^{n+1} = b^n >$.

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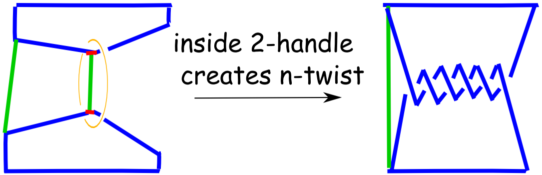
These lie in X_0 so no further effect on $\pi_1(X)$, but:



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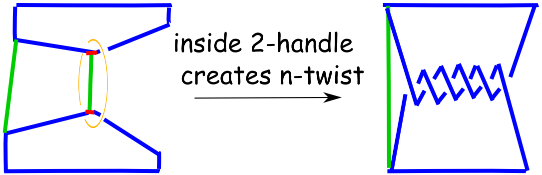

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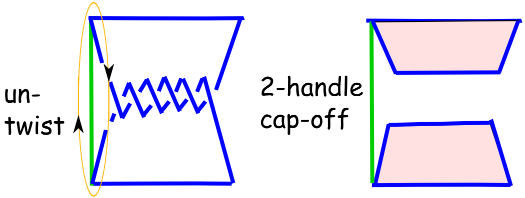

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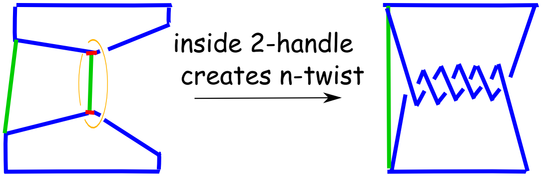

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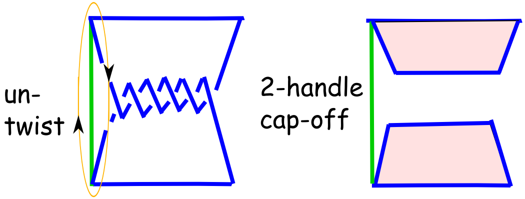

(Bad news: But the structure of ∂X may have changed.)

Reimbedding hope:

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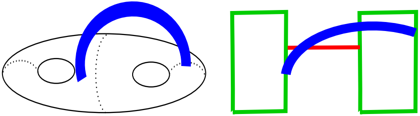

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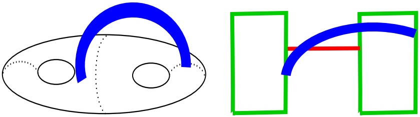
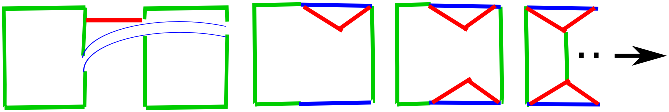


Then complete the construction



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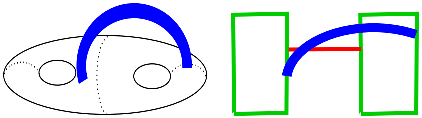

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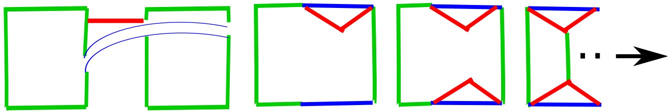

If successful, perhaps reimbedding can substitute for stabilization?