Introduction to Diophantine Conjectures coming from Nevanlinna Theory

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Abstract. This will be a general introductory talk on diophantine conjectures motivated by an analogy with Nevanlinna theory due to C. Osgood, S. Lang, and the speaker. It will recall the basic notation of Nevanlinna theory and corresponding conjectures in diophantine geometry; state the main conjectures; say a few things about the abc conjecture (no major announcements ... sorry); discuss recent work using Schmidt's Subspace Theorem; and mention some ideas about converse statements. Some of the above topics may need to be shortened or omitted, in the interest of setting a good example by ending on time.

§1. Introduction to Diophantine Conjectures ...

- Notation
- Heights, Weil functions, and Nevalinna notation
- Conjectures
- abc
- Future directions

§2. Notation

Throughout this talk –

- k is a number field (or, it could be a function field of dimension 1 over $\mathbb{C}$);
- S is a finite set of places of k containing all of the archimedean places; and
- M_k is the set of all (nontrivial) places of k.

For $v \in M_k$ and $x \in k$, the norm $\|x\|_v$ is

$$\|x\|_v = \begin{cases} |\sigma(x)|^{[k_v:\mathbb{R}]} & \text{if } v \text{ is archimedean,} \\ & \text{corresponding to } \sigma\colon k \to \mathbb{C}\,; \\ (\mathscr{O}_k : \mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)} & \text{if } v \text{ is non-archimedean,} \\ & \text{corresponding to a prime ideal } \mathfrak{p} \ne (0) \\ & \text{of } \mathscr{O}_k, \text{ and } x \ne 0\,. \end{cases}$$

A variety over k is an integral scheme, separated and of finite type over $\operatorname{Spec} k$, and a morphism of varieties over k is a morphism in the category of schemes over k.

A line sheaf is an invertible sheaf (or, informally, a line bundle).

§3. Heights, Weil Functions, and Nevalinna Notation

Heights

There is a unique way (up to $O(1)$, depending on X and $\mathscr{L}$) to assign to each complete variety X over k and each line sheaf $\mathscr{L}$ on X a height $h_{\mathscr{L}}\colon X(\bar{k}) \to \mathbb{R}$ such that:

- **(Additivity)**
$$h_{\mathscr{L}\otimes\mathscr{M}} = h_{\mathscr{L}} + h_{\mathscr{M}} + O_{\mathscr{L},\mathscr{M}}(1)$$
for all varieties X over k and all line sheaves $\mathscr{L}$ and $\mathscr{M}$ on X;
- **(Functoriality)**
$$h_{f^*\mathscr{L}} = h_{\mathscr{L}} \circ f + O_{f,\mathscr{L}}(1)$$
for all morphisms $f\colon X \to X'$ of varieties over k and all line sheaves $\mathscr{L}$ on X'; and
- **(Normalization)**
$$h_{\mathscr{O}(1)}([x_0 : \dots : x_n]) = \frac{1}{[E:k]} \sum_{w \in M_E} \log\max\{\|x_0\|_w, \dots, \|x_n\|_w\} + O_n(1)$$
for all $n \in \mathbb{Z}_{>0}$, all finite extension fields E of k, and all $[x_0 : \dots : x_n] \in \mathbb{P}^n(E)$.

Weil Functions

For a variety X over k, define
$$X(\mathbf{M}) = \coprod_{v \in M_k} X(\bar{k}_v)\ .$$

Then, there is a unique way, up to $O_{X,D}(1)$, to assign to each complete variety X and each Cartier divisor D on X, a function
$$\lambda_D\colon (X \setminus \operatorname{Supp} D)(\mathbf{M}) \to \mathbb{R}$$

such that:

- **(Additivity)**
$$\lambda_{D_1+D_2}(P) = \lambda_{D_1}(P) + \lambda_{D_2}(P) + O_{D_1,D_2}(1)$$
for all $P \in (X \setminus \operatorname{Supp} D_1 \setminus \operatorname{Supp} D_2)(\mathbf{M})$
- **(Functoriality)**
$$\lambda_{f^*D} = \lambda_D \circ f + O_{f,D}(1)$$
for all $f\colon X \to X'$ and all Cartier divisors D on X' such that $\operatorname{Supp} D \not\subseteq \operatorname{im} f$; and

- **(Normalization)**

$$\lambda_{[x_0=0]}([x_0 : \dots : x_n]) = -\frac{1}{[E_w : k_v]} \log \frac{\|x_0\|_w}{\max\{\|x_0\|_w, \dots, \|x_n\|_w\}} + O_n(1)$$

  for all finite E/k, all $v \in M_k$, all $w \in M_E$ with $w \mid v$, all $[x_0 : \dots : x_n] \in \mathbb{P}^n(\overline{E}_w)$, and all $n \in \mathbb{Z}_{>0}$.

In the above, $O(1)$ is zero for all but finitely many $v \in M_k$, where the finite set depends on the same objects as the implicit constant.

Then

$$h_{\mathscr{O}(D)}(P) = \frac{1}{[E : k]} \sum_{w \in M_E} [E_w : k_v] \lambda_D(P_w) + O_D(1)$$

for all finite E/k and all $P \in (X \setminus \operatorname{Supp} D)(E)$.

For X, D, E, w, v, and P as above, define

$$\lambda_{D,w}(P) = [E_w : k_v] \lambda_D(P_w) ,$$

so that

$$h_{\mathscr{O}(D)}(P) = \frac{1}{[E : k]} \sum_{w \in M_E} \lambda_{D,w}(P) + O_D(1) .$$

Nevanlinna Notation

Then we define the proximity function

$$m_S(D, P) = \frac{1}{[E : k]} \sum_{\substack{w \in M_E \\ w \mid S}} \lambda_{D,w}(P)$$

and the counting function

$$N_S(D, P) = \frac{1}{[E : k]} \sum_{\substack{w \in M_E \\ w \nmid S}} \lambda_{D,w}(P) ,$$

so that

$$h_D(P) = m_S(D, P) + N_S(D, P) + O_D(1)$$

and m_S and N_S each satisfy additivity, functoriality, and normalization conditions, as above.

In more concrete terms if $X = \mathbb{P}^1$, $D = [a]$, and $P \in \mathbb{P}^1(k)$ (with $P \neq a, \infty$), then we can use

$$m_S(D,P) = \sum_{v \in S} \max\{0, -\log \|P - a\|_v\}$$

and

$$\begin{aligned} N_S(D,P) &= \sum_{v \notin S} \max\{0, -\log \|P - a\|_v\} \\ &= \sum_{v \notin S} \operatorname{ord}_{\mathfrak{p}}(P - a) \log(\mathscr{O}_K : \mathfrak{p}) \ . \end{aligned}$$

Nevanlinna Theory

The notation in Nevanlinna theory is similar. Let X be a variety over $\mathbb{C}$, let D be a Cartier divisor on X, and let $f\colon \mathbb{C} \to X$ be a holomorphic curve with $f(\mathbb{C}) \nsubseteq \operatorname{Supp} D$. Then the proximity function is defined for $r \in (0, \infty)$ as

$$m_f(D,r) = \int_0^{2\pi} \lambda_D(f(re^{\sqrt{-1}\theta}))\,\frac{d\theta}{2\pi}$$

(where λ_D is a Weil function for D, and can be defined using a smooth metric on $\mathscr{O}(D)$), and the counting function is defined as

$$N_f(D,r) = \sum_{w \in \mathbb{D}_r} \operatorname{ord}_w(f^*D) \cdot \log \frac{r}{|w|} \ .$$

Then

$$T_{D,f}(r) = m_f(D,r) + N_f(D,r) + O(1)$$

is the Nevanlinna height function.

§4. Conjectures

The similarity between Nevanlinna theory and number theory led initially to the following conjecture.

Conjecture 1. *Let X be a smooth projective variety over k, let D be a normal crossings divisor on X (always assumed to be effective and reduced), let $\mathscr{K}$ be the canonical line sheaf on X, let $\mathscr{A}$ be an ample line sheaf on X, and let $\epsilon > 0$. Then there is a proper Zariski-closed subset Z of X, depending only on X, D, $\mathscr{A}$, and ϵ, such that*

$$h_{\mathscr{K}}(P) + m_S(D,P) \le \epsilon\, h_{\mathscr{A}}(P) + O(1) \tag{*}$$

for all $P \in (X \setminus Z)(k)$, where the implied constant in $O(1)$ does not depend on P.

The set Z must depend on $\mathscr{A}$ and ϵ (McKinnon, Levin, Winkelmann, ...)

A Variant for Algebraic Points

Conjecture 2. *Let X, D, $\mathscr{K}$, $\mathscr{A}$, and ϵ be as above, and let $r \in \mathbb{Z}_{>0}$. Then there is a proper closed subset Z of X, depending only on X, D, $\mathscr{A}$, ϵ, and r, such that*

$$h_{\mathscr{K}}(P) + m_S(D,P) \le d_k(P) + \epsilon\, h_{\mathscr{A}}(P) + O(1)$$

for all $P \in (X \setminus Z)(\bar{k})$ with $[k(P):k] \le r$.

Here

$$d_k(P) = \frac{1}{[k(P):k]} \log |D_{k(P)}| \;.$$

The assumption $[k(P):k] \le r$ cannot be omitted (Levin).

Truncated Counting Functions

The inequality (*) can be expressed in a slightly different form as

$$N_S(D,P) \ge h_{\mathscr{K}(D)}(P) - \epsilon\, h_{\mathscr{A}}(P) - O(1) \;.$$

Since non-archimedean norms v are discrete, corresponding Weil functions (for an effective divisor D) can be truncated to their smallest positive value at v. Call this truncated Weil function $\lambda_D^{(1)}$, and define

$$N_S^{(1)}(D,P) = \sum_{v \notin S} \lambda_D^{(1)}(P)$$

for all $P \in (X \setminus \operatorname{Supp} D)(k)$. Clearly $N_S^{(1)}(D,P) \le N_S(D,P) + O(1)$, and therefore Conjecture 1 can be strengthened by replacing (*) with

$$N_S^{(1)}(D,P) \ge h_{\mathscr{K}(D)}(P) - \epsilon\, h_{\mathscr{A}}(P) - O(1) \;.$$

Call this Conjecture 3.

One can also make the same change for Conjecture 2; call this Conjecture 4.

[Draw grid on board]

When $X = \mathbb{P}^1$ and $D = [0] + [1] + [\infty]$, Conjecture 3 is exactly the abc conjecture (Noguchi).

Generic Sets

Definition (Szpiro, Ullmo, Zhang). A set (or sequence) of points in $X(\bar{k})$ is generic if it is infinite and if every infinite subset is Zariski dense.

In Conjectures 1–4, you can omit Z (or, more precisely, take $Z = \operatorname{Supp} D$) if you assert that the inequality holds for a given generic set of rational or algebraic points.

Singularities

In Conjectures 1–4, you can also drop the assumption that D have normal crossings, by adding a term involving the multiplier ideal sheaf.

§5. abc

Conjecture (Masser-Oesterlé). *For each* $\epsilon > 0$ *there is a constant* C *such that the inequality*

$$\max\{|a|,|b|,|c|\} \leq C \prod_{p|abc} p^{1+\epsilon}$$

holds for all relatively prime triples (a,b,c) *of integers such that* $a+b+c=0$.

The abc conjecture would be implied by –

- Conjecture 3 with $X = \mathbb{P}^1$ and $D = [0]+[1]+[\infty]$;
- Conjecture 2 for Fermat curves with $D = 0$; or
- Conjecture 1 for general X, such that $\mathbb{G}_{\mathrm{m}}^{\dim X - 1}$ acts faithfully on $X \setminus D$.

Mochizuki

> "Members of his home university have studied his preparatory papers and waded through his manuscripts on inter-universal Teichmüller theory, communicating with Mochizuki on suggested improvements and adjustments to be made; they plan to give seminars on the material starting in the fall of 2014."

–Greg Martin and Winnie Miao, "abc triples", arXiv:1409.2974, 10 Sept 2014.

§6. Possible Future Directions

Schmidt's Subspace Theorem

Work of Corvaja, Zannier, Evertse, Ferretti, Levin, Ru, ... have led to the following approach.

Embed X into $\mathbb{P}^N$ for large N, such that:

- the image is not contained in any hyperplane, and
- the image of each component of D is contained in a linear subspace of high codimension.

Then apply Schmidt's Subspace Theorem on this $\mathbb{P}^N$, with respect to a set of hyperplanes, some subsets of which have intersection equal to the above-mentioned linear subspace for each component of D.

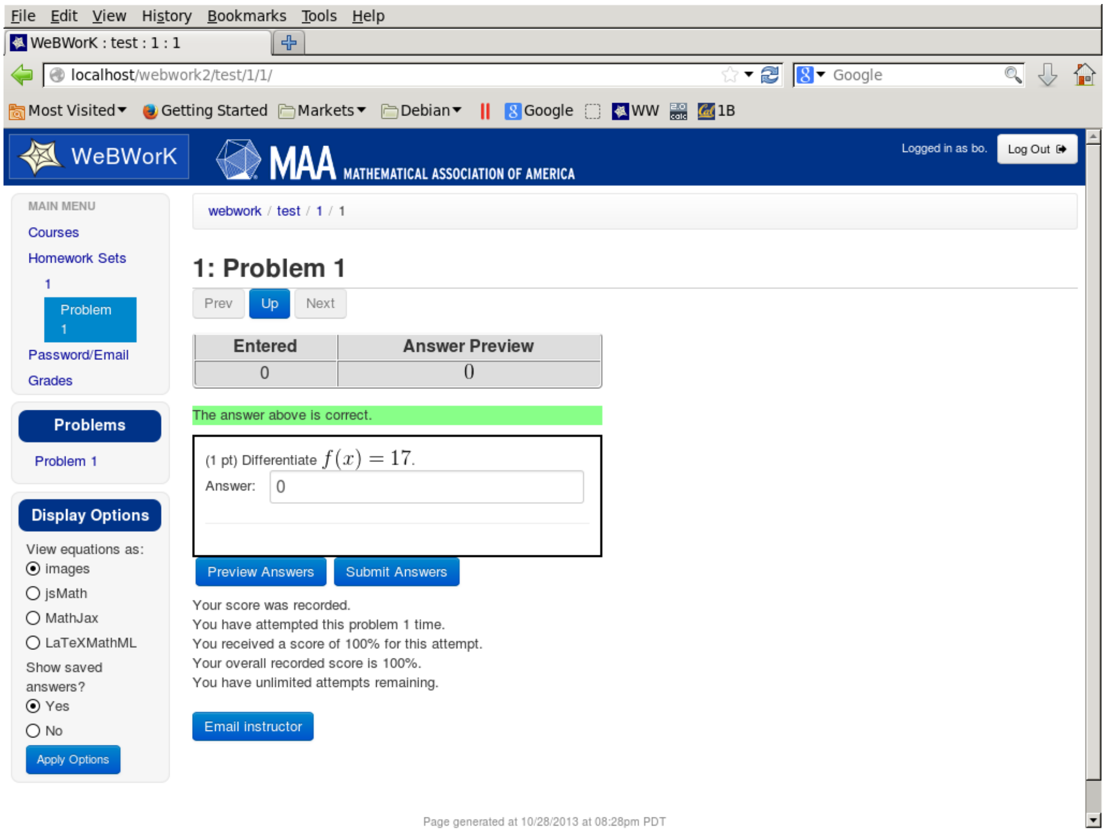
Then Schmidt's Subspace Theorem applied to this set of hyperplanes gives a diophantine inequality for D on X. For more details, see Levin's talk.

Derivatives

By definition, a function is holomorphic if it has a derivative, and this derivative plays a central role in proofs in Nevanlinna theory.

In number theory, what should correspond to the derivative?

We all know what the derivative of an integer is...



In the proof of Schmidt's Subspace Theorem, successive minima play the same role as the associated curves do in the proof of corresponding theorem of Cartan.

Therefore, we make the following conjecture.

Conjecture (Tautological conjecture). *Let X be a smooth projective variety over k with $\dim X > 0$, let D be a normal crossings divisor on X, let $r \in \mathbb{Z}_{>0}$, let $\mathscr{A}$ be an ample line sheaf on X, and let $\epsilon > 0$. Then for all $P \in (X \setminus \operatorname{Supp} D)(\bar{k})$ with $[k(P) : k] \le r$, there is a closed point $P' \in \mathbb{P}(\Omega_{X/k}(\log D))$ lying over P with*

$$h_{\mathscr{O}(1),k}(P') \le N_S^{(1)}(D,P) + d(P) + \epsilon\, h_{A,k}(P) + O(1)\,,$$

where the implicit constant in $O(1)$ depends only on k, S, r, X, D, $\mathscr{A}$, and ϵ.

[Moreover, given a finite collection of dominant rational maps $g_i \colon X \dashrightarrow W_i$ to varieties W_i, there are finite sets Σ_i of closed points on W_i for each i with the following property. For each P as above, P' may be chosen so that, for each i, if P lies in the domain of g_i and if $g_i(P) \notin \Sigma_i$, then P' lies in the domain of the induced rational map $\mathbb{P}(\Omega_{X/k}) \dashrightarrow \mathbb{P}(\Omega_{W_i/k})$.]

Converse Statements?

Is it true that every inequality of the same form as (*), but which is not implied by (*), fails for a Zariski-dense set of rational points (or algebraic points of bounded degree), provided k and S are large enough?

Work of Levin, Ru, ...

Also: one can formulate a "diophantine exceptional set" for integral points; can one show that this coincides with a corresponding "special set" for varieties that are not assumed to be complete?

This leads to the following simple question (simple to state, that is).

Definition. A variety X over a number field k has potential density of integral points if there exist

- a finite extension E of k,
- a finite set S of places of E (containing all of the archimedean places), and
- a model $\mathscr{X}$ for X over $\mathscr{O}_k$,

such that $\mathscr{X}(\mathscr{O}_{E,S})$ is Zariski-dense.

Conjecture. *Let A be a (semi)abelian variety over k, and let Z be a Zariski-closed subset of codimension at least 2. Then $A \setminus Z$ has potential density of integral points.*

Theorem (Hassett and Tschinkel 2001, based on an idea of McKinnon). *Let $G = \prod_{i=1}^{n} G_i$ be a group variety over a number field, where each factor G_i is either $\mathbb{G}_\mathrm{m}$ or a geometrically simple abelian variety. Let Z be a closed subset of G such that $\operatorname{codim} Z > \dim G_i$ for all i. Then $G \setminus Z$ has potential density of integral points.*