Specialization of Linear Series for Algebraic and Tropical Curves (14w5133)

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1 Overview of the Field

Recent developments in tropical geometry have opened the possibility of significant applications to the classical study of linear series and projective embeddings of algebraic curves. These developments take the form of "specialization lemmas" that control how curves with special linear series behave in degenerating families [5, 8]. Here, algebraic curves are 1-dimensional algebraic varieties and linear series are certain vector spaces of functions on these curves. The technique of degeneration involves considering a family of algebraic curves that may become singular and to deduce properties of the smooth members of the family from that of the singular ones. The specialization lemma allows one to use degenerations more pathological than those considered before and to understand the smooth members in terms of the combinatorics of the singular members, specifically in the study of their dual graphs.

Our technical setup is as follows. We fix a curve X and consider a regular, generically smooth, semi-stable curve $\mathcal{X}$ over a discrete valuation ring R with generic fiber X. The special fiber $\mathcal{X}_k$ is a reduced nodal curve over the residue field k of R. For simplicity, we assume k is algebraically closed. We let $\bar{G} = (G, w)$ be the *(weighted) dual graph* of $\mathcal{X}_k$, whose vertices v correspond to irreducible components C_v of the curve $\mathcal{X}_k$ and whose (multi)-edges (u, v) are in one-to-one correspondence with the components of pairwise intersections $C_u \cap C_v$. The weight function $w\colon V(G) \to \mathbb{Z}_{\geq 0}$ associates to every vertex v of G, the geometric genus of the component C_v.

Figure 1: A curve and its dual graph.

Classically, the study of such degenerations was essentially limited to degenerations of “compact type”, where the dual graph of the special fiber is a tree and the Jacobian of the special fiber is compact. Eisenbud and Harris developed their theory of “limit linear series” to describe the linear series on components of the special fiber obtained as limits of special linear series on the general fiber, in degenerations of compact type, and used it to prove many new results in the geometry of curves [28]. Among them, we can mention three contributions: the full symmetric action of monodromy on Weierstrass points of a general curve [29], the fact that M_g is of general type for g at least 24 [27], and new simple proofs of the Brill-Noether and Gieseker-Petri Theorems [26]. The later are the core of classical Brill-Noether theory which studies the behavior of linear series on a general curve, that is curves outside a subset of positive codimension in the moduli space.

Tropical geometry, on the other hand, is most naturally suited to study “maximal degenerations” of curves – the opposite of compact type degenerations – where the dual graph of the special fiber has the largest possible first Betti number (equal to the genus of the generic fiber) and the Jacobian of the special fiber is affine. In its most basic form, the theory of tropical linear series studies the possible ways in which the degree of a special divisor in the generic fiber can be distributed over the components of the special fiber. There is a certain tension between combinatorics and algebraic geometry: when the dual graph of the special fiber has simple combinatorics, the algebraic geometry of the components may be complicated; when the components are simple, the combinatorics of distributing degrees among points of the dual graph becomes complicated and rich. This combinatorial information is surprisingly powerful; it has already led to new proofs of the Brill-Noether Theorem [23] and the Gieseker-Petri Theorem [12, 38]. Similar methods have been used by Castryck–Cools [21], and Kawaguchi [41, 42] to compute the gonality of the generic curve with a given Newton polygon. All of these proofs rely on the original, prototypical specialization lemma, due to Baker [8]. Remarkably, to prove a result about generic curves, it suffices to prove a combinatorial statement for a single suitably-chosen example.

Several more advanced specialization lemmas have been proved since Baker’s original version, allowing non-maximal degenerations and degenerations with singular components, taking into account the genus of components of the special fiber, and permitting the divisors, as well as the curves, to move in families. Most recently, Amini and Baker [4] have used Berkovich’s machinery for nonarchimedean analytic geometry to develop a specialization lemma for “tropical limit linear series.” This new perspective takes into account both the distribution of degrees on components of the special fiber and also the linear series that can appear on these components. Their results apply to semistable curves which are not necessarily of compact type

An important remark is that combinatorial information on dual graphs does not suffice to develop this new enriched theory. For this purpose, Amini-Baker developed the notion of a *metrized complex* of an algebraic curve X which arises from the data of special fiber $\mathcal{X}_k$ together with a metric structure on its dual dgraph. The metric is determined by the speed with which the nodes form in the degeneration. These metrized complexes satisfy a specialization lemma as in [5, 8], and a Riemann–Roch Theorem which generalizes both the classical Riemann-Roch theorem and its graph-theoretic and tropical analogues due to Baker-Norine [10], Gathmann-Kerber [33], and Mikhalkin-Zharkov [53].

In the compact type case, Amini and Baker exactly recover the Eisenbud-Harris theory of limit linear series, while in the maximally degenerate case this new theory is strictly stronger than any previous tropical specialization lemma. These more refined specialization lemmas rely on the tropical concept of lifting. A natural question arises in this context. Given a piece of combinatorial data on the dual graph of a singular curve, for example, a tropical divisor, can we extend it to an algebraic geometric object on a smoothing of the given singular curve? More restrictive conditions on lifts lead to better specialization lemmas.

This workshop brought together leading experts and young researchers from tropical geometry and the classical theory of linear series on algebraic curves. The meeeting provided the first opportunity to bridge these communities and advance the state of the art on both sides. The three primary goals set up for the meeting were achieved:

- **Objective 1** Create an opportunity for tropical geometers and researchers in the classical theory of linear series to learn the state of the art and fundamental techniques on both sides, and foster mutual understanding across these two disciplines.

- **Objective 2.** Explore possibilities for combining classical and tropical techniques to resolve outstanding problems on both sides, such as those mentioned above.

- **Objective 3.** Help advanced graduate students and recent PhDs working in these two areas to connect with peers and experts across these two subjects, and encourage an understanding of the rigorous connections between tropical geometry and classical linear series. Participants included 4 PhD students and 10 postdocs.

As explained above, new specialization lemmas in tropical geometry simultaneously generalize both the classical theory of limit linear series, due to Eisenbud and Harris, as well as the original tropical specialization lemma, due to Baker. Both approaches lead to proofs of the Brill–Noether and Gieseker–Petri Theorems, and the extent to which combining the two approaches may lead to significant new results is not yet known, but experts on both sides are very interested in exploring the possibilities as a result of discussions that occurred during the meeting. The workshop included an evening open problem session presenting the some of the outstanding open questions in these areas, including the Maximal Rank Conjectures and the Kodaira dimension of M_{23}. We discuss these topics in Section 2.

2 Recent Developments and Open Problems

There are a number of open questions in linear systems on algebraic curves that may be approachable by the methods of tropical geometry. Some of these were surveyed by Brian Osserman in his talk. The Maximal Rank Conjecture concerns the behavior of an embedding given by a symmetric power of a linear system. It is phrased in terms of the rank of a particular linear map between symmetric powers of spaces of sections of a line bundle. A refinement of this conjecture, the Strong Maximal Rank Conjecture, predicts the dimension of the locus in the Picard variety of a general curve parametrizing line bundles where these linear maps drop rank. One particular case of this conjecture would imply that the moduli space of curves of genus 23 is of general type. Prescribed ramification involves a series of questions about extending Brill-Noether theory from sections of a line bundle to section of a line bundle with prescribed ramification. Indeed, Brill-Noether theory involves the existence of a linear system with prescribed degree and rank on a generic curve when it is predicted by a dimension count. This can be generalized to ask about the existence of a linear systems with prescribed ramification sequences at particular points. This is known by work of Eisenbud-Harris is characteristic 0, but is open in characteristic p smaller than the degree. One may also want to know how many linear systems there are if the expected dimension is 0. Another variant of these questions is the real case where much is open when the genus is greater than 0.

The tools used in tropical geometry are not completely understood, and there are many open questions regarding their nature. The precise algebraic geometric interpretation of some of the most important tropical invariants of divisors, including the “rank” that appears in Baker’s specialization lemma, remain mysterious. In [17], Caporaso introduced the notion of algebraic rank and conjectured that it behaves as a minimax formula over curves and line bundles. Although the conjecture fails in general by work of Caporaso, Len and Melo [18], it is true in a large number of cases. While there are combinatorially defined total spaces of linear systems and Brill Noether loci, their relationship to their classical analogues is still unexplored. Many open questions in tropical geometry centre on the question of *lifting*, that is, given a combinatorial object can one find a algebraic geometric object that specializes to it. For example, one may try to lift a tropical divisor of a particular rank to a classical divisor of the same rank. This is important in practice as the bound given by the tropical rank function may not be sharp because it is considering divisors that do not lift. In fact, there is a hierarchy of lifting conditions coming from extending divisors to a family. Progress on these issues should give insight into lifting tropical linear systems, allowing refinements to the specialization lemma and further applications to the classical theory.

A particular gap in applying tropical geometry to questions in linear systems stems from an insufficiently-developed theory of ranks of linear maps in tropical geometry. Indeed, a number of conjectures in linear systems such as the maximal rank conjecture or Green’s conjecture are phrased in terms of the ranks of particular linear maps between vector spaces of sections of line bundles. However, there is not a suitable Abelian category in tropical geometry to allow one to talk about kernels and cokernels of linear maps. A recent, important development is the proof of the Geiseker-Petri theorem due to Jensen-Payne [38] which used tropical techniques to show that a particular multiplication map is injective. This was done by using an explicit basis to study the multiplication map. This paper opens the possibility of using tropical techniques to study the ranks of linear maps considered in these open problems.

3 Presentation Highlights

The talks reviewed techniques in the subject, introduced the audience to open problems, and announced new results. Rather than presenting them in chronological order, we group them by topic.

Background material was surveyed to bridge the backgrounds of the participants coming from different research communities during the first morning of the workshop. **Farbod Shokrieh** explained the Baker-Norine theory of linear systems of graphs and its main result: the Riemann–Roch Theorem for graphs and tropical curves. Rather than presenting the original proofs of [10], he gave an alternative proof as in [6] based on the burning algorithm due to Dhar. This approach is more suited for effective computations of these ranks. Finally, he discussed another approach to this result by means of monomial initial ideals, lattice binomial ideals and Alexander Duality developed by Manjunath and Sturmfels in [51].

Melody Chan introduced Baker's specialization lemma as a powerful technique for bounding the rank of a divisor on a curve by using combinatorics of the dual graph of a degeneration of a curve. In addition to providing a proof of this foundational result, she discussed illuminating examples regarding Specialization of Weierstrass points on smooth plane quartics, setting the context for the tropical proof of Brill–Noether Theorem [23].

Brian Osserman surveyed the most relevant open problems in linear series, concentrating on the maximal rank conjecture, prescribed ramification, and real linear series, described in detail in Section 2.

Questions about the nature of the Baker-Norine rank were discussed in talks by Yoav Len, Shu Kawaguchi and Dustin Cartwright. The Baker-Norine rank gives an upper bound for the rank of divisors on curves with particular specialization. One may ask when this bound is sharp. **Shu Kawaguchi** gave a complete answer in the genus 3 and hyperelliptic cases: in these cases he gave necessary and sufficient combinatorial conditions for the upper bound in the specialization lemma to be tight, following [43, 44]. **Dustin Cartwright** presented examples where this bound is never sharp. These examples were constructed using realizability techniques from matroid theory: indeed, by relating these linear systems to combinatorial ones, he relates the lifting question to the realization of certain matroids; interesting counterexamples come from non-representable matroids. **Yoav Len** discussed the notion of algebraic rank of divisors on tropical curves as defined by Caporaso in [17]. It is defined in terms of a minimax formula in terms of curves and divisors with particular specialization. Although this notion differs from the Baker-Norine rank, it has many desirable properties. In particular, his lecture presented interesting examples where both ranks differ, as shown in [18].

A new application of linear systems on graphs to linear systems was discussed by **David Jensen**. His presentation was twofold. On one hand, he surveyed the tropical perspective of Brill–Noether theory and presented tropical proofs of the Brill–Noether theorem, following [23]. Second, he discussed tropical multiplication maps and gave a tropical criterion for a curve over a valued field to be Gieseker–Petri general, thus providing a tropical proof of the Gieseker–Petri Theorem, as in [38]. Significantly, this proof is able to use linear algebraic genericity techniques that have previously resisted translation into tropical geometry.

Extensions of line bundles to degenerate curves were discussed by Jesse Kass and Eduardo Esteves. Notably, these questions are nontrivial because the naïve moduli spaces of line bundles fail to be compact when the dual graph of the degeneration is not a tree. One must expand the moduli spaces to include some non-classical objects. **Jesse Kass** compared two candidate compactifications, the Néron model and the moduli space of stable sheaves proving that they are the same [**?**].

Eduardo Estevez presented work in progress on new perspectives for limit linear series, attempting to merge previous approaches to construct a variety of limit linear series. First, Eisenbud–Harris work for curves of compact type [27, 28]. Second, L. Caporaso's approach by construction a relative compactification $\overline{P^d_g}$ of M_g of the relative Jacobian. Third, L. Maino's construction of a moduli space E_g of enriched curves over $\overline{M_g}$ that captured the essence of the inseparability of the relative Jacobian [50]. And fourth, Osserman's approach to the subject that incorporated the data coming from degenerations of linear series in order to construct meaningful varieties $G^r_d(X)$ of limit linear series for two-component curves X of compact type [55]. The goal of his talk was to build a projective variety over $\overline{P^d_g}$ whose fibers parameterize Maino-style enriched structures and their degenerations, and whose general points parameterize Osserman-style limit linear series.

Applications of tropical geometry to studying moduli problems were discussed by Renzo Cavalieri and Dan Abramovich. **Renzo Cavalieri** discussed Hurwitz numbers which are weighted counts of branched covers of curves. He related the piecewise polynomiality of Hurwitz numbers, a phenomenon proved by Goulden-Jackson-Vakil in [35] to wall-crossing in the enumeration of tropical covers of curves, as proven in [13]. **Dan Abramovich** discussed tropical geometry and skeletons of analytic spaces associated to toroidal embeddings (in the sense of Thuillier [56]) as basic techniques for constructing combinatorial models of moduli problems from a given stratification of the input spaces. In addition to presenting joint work with Caporaso and Payne on the tropicalization of the moduli space of curves, he presented work in progress on how to address the well-known phenomenon of superabundance[1] of tropical curves described in [52], by studying the moduli space of log stable maps from the tropical perspective.

Another topic discussed at the workshop was the application of specialization lemmas and tropical geometry to number theory. **David Zureick-Brown** explained his recent results with Eric Katz [40] on the Chabauty-Coleman method for bounding the number of rational points on curves of low Mordell-Weil rank. When the curves are of bad reduction at a given prime, one may obtain sharper bounds by using the theory of linear series on metrized complexes of curves. **Janne Kool** presented her recent work with Gunther Cornelissen and Fumiharu Kato on the combinatorial Li-Yau inequalities [25]. In the classical case, these inequalities relate the degree of a map with the smallest nonzero eigenvalue of the Laplace operator and the volume of the curve [47]. Using the Laplacians of the dual graph they provide bounds the gonality of a curve over a given non-Archimedean field. This in turn yields a finiteness statement for the number of rational points of low degree. She concluded her talk by presenting applications to Drinfeld modular curves in analogy with Dan Abramovich's results lower bounds on the gonality of modular curves in the classical case [1].

Filip Cools gave an overview talk on combinatorial methods applied to gonality of curves that are general with respect Newton polygons, surveying many of the results from [21]. By working with linear pencils that are visible from the geometry of the Newton polygon, he explained how this gonality as well as Clifford's index are encoded. As a further application of this method, he presented a concrete way to write down generators of the canonical ideal of the curve, and suggested an approach to producing potential counterexamples to Green's Conjecture, using this result. He explained that, with 7 exceptions, every gonality pencil on (the smooth projective model of) a curve that is general with respect to Newton polygon is combinatorial, a result obtained independently by Kawaguchi [41, 42] and Castryck–Cools [21]. He also explained the 7 exceptions in detail, and computed the gonality in each case. He furthermore presented lower bounds on the gonality of using near-gonality pencils and presented explicit formulas for the Clifford index and the Clifford dimension, as in [21, 41, 42]

Marc Coppens presented the main result of [24], namely Clifford Theorem for metric graphs. He gave a complete, detailed proof of this important result, based on the Riemann–Roch Theorem for graphs, the Abel-Jacobi map for tropical curves [53] and the algorithm for computing rank of divisors on tropical curves given in [6].

Omid Amini discussed work in progress on the limit behaviour of Weierstrass points when specializing from a curve to a graph. He studied a degenerating family of curves equipped with a line bundle and asked how the Weierstrass points of powers of the line bundle specialize to the central fiber of the family. His result can be viewed as the non-Archimedean analogue of a theorem of Mumford-Neeman [54]: he showed that such points are equidistributed with respect to Zhang's measure on the dual graph.

4 Outcome of the Meeting and Scientific Progress Made

As mentioned in Section 1, this conference brought together leading experts and young researchers from two communities that seldom have the opportunity to discuss new ideas and learn from each other. Having a mixture of Ph.D. students, several postdocs and researchers in tropical geometry and the classical theory of linear series on algebraic curves from Europe, Japan and North America provided an ideal environment for exchanging perspectives and start new collaborations.

The overall impression was that this was one of the best conferences ever attended, for most of the participants. The open problem session, together with the discussions and collaborations it initiated throughout the week proved to be one of the most successful activities of the conference. The ample time between the lectures was used by smaller groups to give spontaneous presentations and to *do* mathematics together.

Intangible benefits, such as building a community, establishing mathematical as well as personal connections, and of course, disseminating knowledge, were widely mentioned. We also got feedback from several

[1] A map $f : C \to \mathbb{TP}^n$ is called *superabundant* if the dimension of the space of deformations of the pair (C, f) is strictly larger than that predicted by naïve dimension counts.

participants about very concrete results from the workshop. These range from new projects conceived at the conference to existing projects completed and papers made ready for publication. Here is a sample of such results.

1. Matt Baker, Yoav Len and Nathan Pflueger were able to take steps towards completing an ongoing research project on bitangents of tropical plane quartic curves. The paper was posted shortly after the end of the BIRS workshop as `arXiv:1404.7568`.

2. M. Angelica Cueto and Martin Ulirsch had a long discussion on how to relate non-Archimedean skeleta of toroidal embeddings to geometric tropicalization, and are currently planning to collaborate in order to work out this connection in detail.

3. Martin Ulirsch discussed with Dustin Cartwright, Eric Katz, and Sam Payne several aspects of a current joint project with M.-W. Cheung, L. Fantini and J. Park on the faithful realizability of tropical curves. This project started as a group project in a Mathematical Research Community Summer School in 2013, under the guidance of Sam Payne. Thanks to the discussion he had at BIRS (especially conversations with D. Cartwright) he came around on how to resolve the missing step in one of the main results of the aforementioned project.

4. Filip Cools and Jan Draisma have started collaborating on a paper in which they aim to prove that the locus of metric graphs of (tree) gonality less than expected has exactly the same codimension as in the classical case.

5. Jan Draisma and Anton Leykin started working on signature-based Gröbner bases in the context of equivariant Gröbner bases. By now, they have settled the relevant Noetherianity of first syzygies, in collaboration with Robert Krone.

6. Marc Coppens, Shu Kawaguchi and Kazuhiko Yamaki started collaborating on a lifting of divisors and gonalities, two ideas that grew out of Kawaguchi's presentation.

We end this report with some of the testimonies gathered after the meeting (in alphabetical order):

Marc Coppens said: "I had a very good time at the conference in BIRS and it was very helpful to me. Since I am not attending conferences that often, it was a nice occasion for me to meet some people whose work I have studied and/or used in the past. As an older participant, the conference gave me a new perspective on linear systems on curves. I am especially interested in both the similarities and the differences between linear systems on curves and on graphs and I would like to find out how the study of linear system on graphs can give new information on linear systems on smooth complex and/or real curves. A lot of talks were extremely useful from that point of view. In addition, I had several conversations with Shu Kawaguchi and Kazuhiko Yamaki related to the topic of the talk of Kawaguchi. They asked me some relevant questions on linear systems on algebraic curves to aim at generalizing their results on the subject. This inspired me to search for families of graphs could manifest such generalizations. During the conference, Sam Payne already gave an example of a trigonal graph having infinitely many trigonal pencils. Thus, generalizing their results to all trigonal graphs is impossible, but it might still be achievable for hyperelliptic graphs. By considering these graphs I also found that a lot of nice behaviour of linear systems on curves does not hold for other types of graphs.

After a conversation with Dave Jensen and Sam Payne, I learned that a way to define a weak form of adding a fixed point to a divisor on a graph does not seem to work (this is a pity of course). From Spencer Backman I learned a new way to describe the rank of a divisor on a metric graph using orientations and allowed changes of them. This gives rise to another proof of the Riemann–Roch Theorem for graphs. He also explained how to use this method to try to obtain a more conceptual proof for the complete Clifford Theorem for curves, much more complete that the one I presented in my talk. I started discussing with him and some other participants regarding generalizations of Clifford's Theorem known in curve theory and not yet known for graphs.

In Melody Chan's talk there was a very easy but interesting example showing differences between ranks on the graph and the curve in a degeneration that is due to the fact that in a dual graph one loses information about how components intersect. A Ph.D. student of Filip Cools and myself is currently investigating similarities between results concerning gonality of curves with plane models and the gonality of natural corresponding graphs. In her talk, Chan also observed a similar phenomenon.

From conversations with Matthew Baker I learned that he is currently developping a very refined definition of limit linear pencils on metrized complexes of curves. The aim of this project is to solve the differences that occur in the example given in Chan's talk. My hope is that this refined definition will help solves the differences occuring in the work of my student. If this were the case, then it would indicate that the similarity between the situation of those curves with plane models and the one corresponding to metrized complexes of curves (with rational curves at the vertices) is better observed than by just using metric graphs.

Also during the conference (probably caused by the "atmosphere") I got some new idea for obtaining new results on the gonality of interesting special types of curves using degenerations to graphs. I hope to find time in the near future to think more intensively on it."

Ethan Cotterill said: "I would like to remark five interesting exchange of ideas I had at the BIRS workshop. First, with Eric Katz and Sam Payne, we discussed the problem of characterizing the tropicalizations of singular rational curves; note that this is a variation on a theme already studied by Alicia Dickenstein, Hannah Markwig, Evgeny Shustin, and Luis Tabera (amongst others) in the context of hypersurfaces. Second, Nathan Pflueger explained to me the content of his thesis, in which he studies (the codimension of) Weierstrass-type loci inside the moduli space of curves. Codimension is predicted to be computed by the weight of the associated semigroup, but in many instances this fails and one needs to produce a substitute combinatorial invariant, which he calls the effective weight. In joing work with L.F. Abrantes and R.V. Martins I have been studying how to bound the codimension of (embedded) singular rational curves according to the semigroups of their singularities. Again the codimension is expected to be computed by the weight. It would be interesting to see how far this analogy can be pushed.

Third, Spencer Backman explained to me how to view divisors on a (metric) graph as partial orientations, and how to compute the rank of these according to this point of view. Fourth, Diane Maclagan explained to me her recent work with Felipe Rincon, in which they make the scheme-theoretic tropicalization due to Noah and Jeffrey Giansiracusa effective. It seems this has good potential for proving liftability/realization results for tropical subvarieties, e.g. curves in surfaces or lines in threefolds. Finally, Eric Katz explained to me the basic philosophy underlying log geometry, and how it serves as a tool for proving correspondence theorems in tropical geometry."

Jan Draisma said: "In addition to starting two new research projects with other participants of this BIRS workshop, I had been many stimulating discussions with M. Angelica Cueto (on faithful tropicalizations), with Gunther Cornelissen and with Sam Payne. Finally, I had interesting conversation with Janne Kool, in which I encourage her to apply for a postdoctoral position in the Networks program in the Netherlands."

Eduardo Esteves said: "Thank you very much for having organized the meeting. It was very interesting for me. I had discussions with Jesse Kass about his approach to proving autoduality for compactified Jacobians by means of the Néron model interpretation. I enjoyed discussing with Filippo Viviani about the approach to limit linear series I presented in my talk. And most of all I profited a lot from discussing with Omid Amini and comparing his tropical approach to limit linear series against my joint project on limit linear series with Brian Osserman."

Jesse Kass said: "The workshop was a great opportunity to meet with researchers, both in my field and in neighboring fields. I had the opportunity to discuss recent work of mine on autoduality with Filippo Viviani and Eduardo Esteves, two people who have done important work on the topic. The feedback they provided was very helpful! Listening to Renzo Cavalieri's talk gave me some new ideas for a project on intersection theory that I have been working with Nicola Pagani. Renzo and I had many interesting discussions at the workshop, which I hope will advance both our research programs."

Nathan Pflueger said: "I spoke at length with Ethan Cotterill about different problems we are both studying concerning numerical semigroups in algebraic geometry. We found that we both have unpublished ideas that could help the other's research, and I will very likely travel to meet with him and collaborate in the following year. In addition, the methods Dave Jensen described in his talk made me realize that some of the techniques I have developed to study classical limit linear series could be very useful in studying the Brill–Noether theory of tropical curves. This led to many fruitful discussions with Sam Payne and David Jensen about further directions of work, which could have significant applications for both tropical and algebraic curves. "

Martin Ulirsch said: "As a graduate student I immensely profited from being able to discuss my research projects with established and junior researchers in this workshop."

Ravi Vakil said: "I came in the hopes of making new connections. And as a result, I think I may embark on a whole new theme of research (with Matt Baker). So I am very glad to have come."

Kazuhiko Yamaki said: "I had conversations with several participants about trigonal curves and trigonal graphs among other things. One of these discusions gives me a nice perspective about rank-preseving liftability of divisors from graphs to curves. This would contribute to my research on the relation between graphs and curves. Another discussion enabled me with new techniques to find interesting examples of graphs and curves in which the rank of divisors might behaves well, a significantly new viewpoint that I have never found before. Even after the workshop, I am in correspondence with some of the participants on lifting of divisors, gonalities, and so on. This workshop will certainly enrich my future research."

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