Simple homogeneous structures

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Introduction

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The study of **simple theories/structures** has developed, via *stability theory*, from Shelah's *classification theory* of complete first-order theories and their models.

A central tool in this context is a sufficiently well behaved notion of **independence**.

I will present some results in the intersection of these areas, i.e. we consider structures that are **both simple and homogeneous**. References (containing more references) follow at the end.

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2. Every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.
3. $\mathcal{M}$ is the Fraïssé limit of an *amalgamation class*.

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Via the Engeler, Ryll-Nardzewski, Svenonious characterization of ω-categorical theories: *every infinite homogeneous structure has ω-categorical complete theory.*

Classifications of some homogeneous structures

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The following classes of structures, to mention a few, have been *classified*:

1. homogeneous partial orders (Schmerl 1979).
2. homogeneous (undirected) graphs (Gardiner; Golfand, Klin; Sheehan; Lachlan, Woodrow; 1974–1980).
3. homogeneous tournaments (Lachlan 1984).
4. homogeneous directed graphs (Cherlin 1998).
5. infinite homogeneous stable V-structures for any finite relational vocabulary V (Lachlan, Cherlin, Knight... 80ies).
6. homogeneous multipartite graphs (Jenkinson, Truss, Seidel 2012).

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Some concepts, leading to the notion of SU-rank:
Suppose that T is simple and $\mathcal{M} \models T$. If $A, B, C \subseteq M$ then

"A is **independent** from B over C" $\iff A \underset{C}{\downarrow} B \iff$

$\mathrm{tp}(\bar{a}/B \cup C)$ does not **divide/fork** over C, for every $\bar{a} \in A^n$ and $n < \omega$.

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Suppose that $A \subseteq M \models T$ and let $p(\bar{x})$ be a complete type over A.
$\mathrm{SU}(p) \geq 0$,
$\mathrm{SU}(p) \geq \alpha + 1$ if there is a complete extension $q(\bar{x}) \supseteq p(\bar{x})$ (over some $B \supset A$) such that $\mathrm{SU}(q) \geq \alpha$ and q divides/forks over A.
$\mathrm{SU}(p) = \alpha$ if $\mathrm{SU}(p) \geq \alpha$ and $\mathrm{SU}(p) \not\geq \alpha + 1$.

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The **SU-rank of T**, as well as of every $\mathcal{M} \models T$, is the supremum (if it exists) of the SU-ranks of all 1-types over $\emptyset$ of T.

If the SU-rank of T is finite, then (by the Lascar inequalities) $\mathrm{SU}(p) < \omega$ for every type p of T of any arity.

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Binary vocabulary: a finite relational vocabulary in which all symbols have arity ≤ 2.

Binary structure: a V-structure for some binary vocabulary V.

Examples

The following structures are ω-**categorical and supersimple with finite rank**.

1. *vector space over a finite field*.
2. ω*-categorical superstable structures* (where *superstable* = *supersimple* + *stable*).
3. *smoothly approximable structures* (which contains the previous class but need not be stable).
4. *random graph/structure* has rank 1 and is homogeneous.
5. *random k-partite graph/structure* ($k < \omega$) has rank 1 and can be made homogeneous by expanding with a binary relation.
6. the *line graph of an infinite complete graph* has rank 2 and can be made homogeneous by expanding with a ternary relation.

More examples

1. Suppose that $\mathcal{R}$ is a binary random structure. Let $n < \omega$ and let $\mathcal{M}$ be a binary structure with universe $M = R^n$ such that for all $\bar{a}, \bar{a}', \bar{b}, \bar{b}' \in M$ $\mathrm{tp}^{at}_{\mathcal{M}}(\bar{a}, \bar{a}') = \mathrm{tp}^{at}_{\mathcal{M}}(\bar{b}, \bar{b}')$ if and only if $\mathrm{tp}_{\mathcal{R}}(\bar{a}\bar{a}') = \mathrm{tp}_{\mathcal{R}}(\bar{b}\bar{b}')$. Then $\mathcal{M}$ is homogeneous and supersimple with rank n.
2. For any $n < \omega$ one can construct a homogeneous supersimple structure with rank n by "nesting" n different equivalence relations (with infinitely many infinite classes).
3. The previous two ways of constructing (binary) homogeneous supersimple structures of finite rank can be combined to produce more complex examples.

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More generally we have:

Theorem 2 [K1] *Suppose that $\mathcal{M}$ is a structure which is binary, simple and homogeneous. Then $\mathcal{M}$ is* **super***simple with* **finite** *rank (which is bounded by the number of 2-types over $\emptyset$).*

Theorem 2: rough proof sketch (1st page of 4)

Suppose that $\mathcal{M}$ is binary, simple and homogeneous.
Let $T = Th(\mathcal{M})$ and $t = |S_2(T)|$ = the number of 2-types over $\emptyset$ w.r.t. T.
Let $S_2(T) = \{p_1(x, y), \dots, p_t(x, y)\}$.
Without loss of generality we may assume that every $p_i(x, y)$ is *isolated* by an *atomic* formula $R_i(x, y)$.

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It suffices to show that for every $\mathcal{N} \models T$ there do **not** exist $a \in N$ and *finite* sets $\emptyset = B_0 \subset B_1 \subset \ldots \subset B_{t+1} \subset N^{\mathrm{eq}}$ such that $tp(a/B_{n+1})$ divides over B_n for every $n \leq t$.

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First prove that *it suffices to show that there do* **not** *exist $a \in M$ and* finite *sets $\emptyset = B_0 \subset B_1 \subset \ldots \subset B_{t+1} \subset M$ such that $tp(a/B_{n+1})$ divides over B_n for every $n \leq t$.*

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Then, towards a contradiction, assume there are $a \in M$ and *finite* sets $\emptyset = B_0 \subset B_1 \subset \ldots \subset B_{t+1} \subset M$ such that $tp(a/B_{n+1})$ divides over B_n for every $n \leq t$.

Definition: If $\mathcal{N}$ is a *simple structure* and R a *binary relation symbol* of its vocabulary, then we call R a **nondividing relation w.r.t. to** $\mathcal{N}$ if for all $c, d \in N$, $\mathcal{N} \models R(c, d) \implies c \mathop{\smile\!\!\!\!\!\!\,|}^{\mathcal{N}} d$, where $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathcal{N}}$ denotes independence with respect to $\mathcal{N}$.
R is a **dividing relation w.r.t.** $\mathcal{N}$ if for all $c, d \in N$,
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Recall that $R_1(x, y), \ldots, R_t(x, y)$ isolate $p_1(x, y), \ldots, p_t(x, y)$, respectively, which enumerate $S_2(T)$.

The aim is to show that

each one of $R_1, \ldots, R_t$ *is a dividing relation w.r.t.* $\mathcal{M}$.

But this contradicts the fact that there are $c, d \in M$ such that $c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}} d$ and $\mathrm{tp}_{\mathcal{M}}(c, d)$ is one of $p_1(x, y), \ldots, p_t(x, y)$.

To achieve this (contradiction) we find substructures $\mathcal{M}_{t+1} \subset M_t \subset \ldots \subset \mathcal{M}_1 \subset \mathcal{M}_0 = \mathcal{M}$ for which we can prove the following:

1. For all $n \leq t$, $\mathcal{M}_n$ is infinite, simple and homogeneous.
2. For all $n \leq t$ and all $c, d \in M_n$, $c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}_n} d \iff c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathcal{M}}_{B_n} d$.
3. For all $0 < n \leq t$, there is $0 < i \leq t$ such that R_i is a nondividing relation w.r.t. $\mathcal{M}_n$ and, for all $k < n$, R_i is a dividing relation w.r.t. $\mathcal{M}_k$.

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Now (3) implies that **every** $R_1, \ldots, R_t$ is a dividing relation w.r.t. $\mathcal{M}_0 = \mathcal{M}$ (this gives the contradiction).

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Remark: The proof of (3) uses that if $\mathcal{N}$ is simple and ω-categorical, $\bar{c} \in N^m$ for some $m < \omega$ and $D \subseteq N$ is *finite*, then the **preweight of** $tp_{\mathcal{N}}(\bar{c}/D)$ is finite; a result due to D. Palacin [P].

Theorem 2: rough proof sketch (4th and last page)

The structures $\mathcal{M}_{t+1} \subset M_t \subset \ldots \subset \mathcal{M}_1 \subset \mathcal{M}_0 = \mathcal{M}$ are defined as follows:

$\mathcal{M}_n$ is the substructure of $\mathcal{M}$ with universe

$M_n = \{a' \in M : \mathrm{tp}_{\mathcal{M}}(a'/B_n) = tp_{\mathcal{M}}(a/B_n)\}$

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$\emptyset = B_0 \subset B_1 \subset \ldots B_{t+1} \subset M$ exist such that $tp(a/B_{n+1})$ divides over B_n for every $n \leq t$.

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This ends the proof sketch of Theorem 2.

Triviality and 1-basedness

Let T be simple.

T, as well as every $\mathcal{M} \models T$, has **trivial dependence** if whenever $\mathcal{M} \models T$, $A, B, C \subseteq \mathcal{M}^{\mathrm{eq}}$ ($\mathcal{M}$ extended by imaginaries) and $A \not\mathop{\smile\!\!\!\!\!\mid}_C B$, then there is $b \in B$ such that $A \not\mathop{\smile\!\!\!\!\!\mid}_C \{b\}$.

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T, as well as every $\mathcal{M} \models T$, is **1-based** if and only if for all $\mathcal{M} \models T$ and all $A, B \subseteq \mathcal{M}^{\mathrm{eq}}$ we have $A \downarrow_C B$ where $C = \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A) \cap \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B)$. ('$\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}$' is the algebraic closure in $\mathcal{M}^{\mathrm{eq}}$)

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Examples of 1-based structures: all previously given examples.

In fact, **all known examples of simple homogeneous structures** are *super*simple with *finite rank*, *1-based* and have *trivial dependence*.

Subsets of rank 1

Suppose that $\mathcal{M}$ is simple and $A \subseteq M^{\mathrm{eq}}$ is definable by a formula with parameters from C. I say that A **has rank 1** (with respect to C) if for every $a \in A$, $tp(a/C)$ has rank 1.

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If $A \subseteq M^{\mathrm{eq}}$ is C-definable and has rank 1, then (A, cl) is a **pregeometry** if

$$\mathrm{cl}(B) = \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B \cup C) \cap A \text{ for every } B \subseteq A.$$

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The notions of **trivial dependence and 1-basedness can be "relativized" to to definable sets of rank 1.**

For example, if (A, cl) is as above, then we call it **trivial** if whenever $B \subseteq A$ and $a \in \mathrm{cl}(B)$ then $a \in \mathrm{cl}(b)$ for some $b \in B$.

Usefulness of rank 1 sets

By combining results of Macpherson, Hart–Kim–Pillay and De Piro–Kim, we get:

Fact: *Suppose that $\mathcal{M}$ is ω-categorical and supersimple with finite rank. Then:*

1. *$\mathcal{M}$ is 1-based $\iff$ every definable subset of $\mathcal{M}^{\mathrm{eq}}$ with rank 1 is 1-based.*
2. *$\mathcal{M}$ has trivial dependence $\iff$ every definable pregeometry in $\mathcal{M}^{\mathrm{eq}}$ is trivial.*
3. *If $\mathcal{M}$ is* **homogeneous** *then $\mathcal{M}$ is 1-based if and only if $\mathcal{M}$ has trivial dependence.*

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2. *$\mathcal{M}$ has trivial dependence $\iff$ every definable pregeometry in $\mathcal{M}^{\mathrm{eq}}$ is trivial.*
3. *If $\mathcal{M}$ is* **homogeneous** *then $\mathcal{M}$ is 1-based if and only if $\mathcal{M}$ has trivial dependence.*

Hence, for supersimple homogeneous structures $\mathcal{M}$ with finite rank it suffices to understand 1-basedness/triviality for their definable rank 1 subsets (of M^{eq}).

Implications of 1-basedness and/or trivial dependence

We now look at a couple of results which give better understanding of the "fine structure" of binary homogeneous simple structures that are 1-based and/or have trivial dependence.

First we consider the consequences of trivial dependence on the (from $\mathcal{M}^{\text{eq}}$) "inherited" structure on subsets of rank 1.

Then we investigate (in particular primitive) binary homogeneous 1-based structures.

Binary random structures

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$\mathcal{M}$ is a **binary random structure** if it does **not** have a minimal forbidden configuration of cardinality ≥ 3.

Example: random graph/structure.
Nonexample: random bipartite graph (odd cycles are forbidden).

Canonically embedded structures

Roughly speaking: $\mathcal{M}^{\mathrm{eq}}$ is the extension of $\mathcal{M}$ with **imaginary elements**, i.e. elements that correspond to equivalence classes of $\emptyset$-definable equivalence relations on M^n (for $0 < n < \omega$).

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The **canonically embedded structure of $\mathcal{M}^{\mathrm{eq}}$ over A with universe C** is the structure $\mathcal{C}$ which, for every $0 < n < \omega$ and A-definable relation $R \subseteq C^n$, has a relation symbol which is interpreted as R (and $\mathcal{C}$ has no other symbols).

Note that for all $0 < n < \omega$ and all $R \subseteq C^n$,
R is $\emptyset$-definable in $\mathcal{C} \iff R$ is A-definable in $\mathcal{M}^{\mathrm{eq}}$.

Canonically embedded structures with rank 1

Let $\mathcal{M}$ and $\mathcal{N}$ be two structures which need **not** necessarily have the same vocabulary.

$\mathcal{N}$ is a **reduct** of $\mathcal{M}$ if $M = N$ and for all $0 < n < \omega$ and all $R \subseteq M^n$: R is $\emptyset$-definable in $\mathcal{N} \implies R$ is $\emptyset$-definable in $\mathcal{M}$.

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Theorem 3. [AK] *Suppose that $\mathcal{M}$ is a binary, homogeneous, simple structure with trivial dependence. Let $A \subseteq M$ be finite and suppose that $C \subseteq M^{\mathrm{eq}}$ is A-definable and only finitely many 1-types over $\emptyset$ are realized in C. Assume that $\mathrm{tp}(c/A)$ has rank 1 for every $c \in C$. Let $\mathcal{C}$ be the canonically embedded structure of $\mathcal{M}^{\mathrm{eq}}$ over A with universe C.* **Then $\mathcal{C}$ is a reduct of a binary random structure.**

Homogeneous 1-based structures

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Remark: Suppose that $\mathcal{M}$ is homogeneous and has a nontrivial equivalence relation $E \subseteq M^2$. Let N be one of the E-classes. Then the substructure of $\mathcal{M}$ with universe N is homogeneous. All simplicity theoretic properties considered also transfer from $\mathcal{M}$ to $\mathcal{N}$.

Homogeneous 1-based structures (cont.)

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Theorem 4. [K2] *Suppose that $\mathcal{M}$ is a binary, primitive, homogeneous, simple and 1-based structure. Then $\mathcal{M}$ is strongly interpretable in a binary random structure.*

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Theorem 4. [K2] *Suppose that $\mathcal{M}$ is a binary, primitive, homogeneous, simple and 1-based structure. Then $\mathcal{M}$ is strongly interpretable in a binary random structure.*

Remark: Theorem 4 can be generalized by replacing *primitivity* with *"height 1"*, meaning that there is a $\emptyset$-definable $C \subseteq M^{\mathrm{eq}}$ of rank 1 such that $M \subseteq \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(C)$.

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4. Is every (binary) primitive simple homogeneous structure a (binary) random structure?

5. Can all (binary) simple homogeneous structures be classified?

Problems (cont.)

6. Can some of the results obtained so far be generalised to some homogeneous *non*simple structures?
For example, the class of **rosy** structures have a notion of independence and an analogoue of SU-rank. However, rosy *non*simple structures do *not* satisfy the independence theorem, which has been heavily used (together with binarity) in the mentioned results.

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Question: What is the relevance of simple homogeneous structures with regard to constraint satisfaction, Ramsey theory or permutation group theory?

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Question: What is the relevance of simple homogeneous structures with regard to constraint satisfaction, Ramsey theory or permutation group theory?

Thanks for listening!

References

[AL] Andrés Aranda López, *Omega-categorical Simple Theories*, Ph.D. thesis, University of Leeds, 2014.

[P] D. Palacín, *On omega-categorical simple theories*, Archive for Mathematical Logic, Vol. 51 (2012) 709–717.

The following articles can be found via the link
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[AK] Ove Ahlman, V. Koponen, *On sets with rank one in simple homogeneous structures*, to appear in Fundamenta Mathematicae.

[K1] V. Koponen, *Binary simple homogeneous structures are supersimple with finite rank*.

[K2] V. Koponen, *Homogeneous 1-based structures and interpretability in random structures*.

More references are found in the sources above.