Distance Constraint Satisfaction Problems

Beyond the ω-categorical...

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The Constraint Satisfaction Problem CSP($\mathcal{B}$) takes as input a *primitive positive* (pp) sentence Φ, i.e. of the form

$$\exists v_1 \ldots v_j \ \phi(v_1, \ldots, v_j),$$

where ϕ is a conjunction of atoms, and asks whether $\mathcal{B} \models \Phi$.

This is equivalent to the Homomorphism Problem – has $\mathcal{A}$ a homomorphism to $\mathcal{B}$?

The structure $\mathcal{B}$ is known as the template...

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All natural finite CSPs have been classified for complexity.

It is conjectured that finite CSPs are all either in P or are NP-complete. Great swathes are classified – undirected graphs, smooth digraphs, 2-domains, 3-domains, conservative languages... Some pathological boundary cases remain unclassified – of interest only to those who are attempting to classify them.

There is a myriad of interesting infinite-domain CSPs whose complexity is unknown – of interest to all in Computer Science.

Classifications for infinite CSPs

P versus NP-complete dichotomies for

- Allen's Interval Algebra (Jeavons-Jonsson-Krokhin).
- fo-definitions in $(\mathbb{Q}; <)$ (Bodirsky-Kára).
- fo-definitions in $\mathcal{RG}$ (Bodirsky-Pinsker).
- fo-expansions of $(\mathcal{R}; +, 1, \leq)$ (Bodirsky-Jonsson-von Oertzen).
- fo-expansions of $(\mathcal{R}; +, 1)$ using $\leq$ (Jonsson-Thapper).
- bounded-degree fo-definitions in $(\mathbb{Z}; \mathit{succ})$ (Bodirsky-Dalmau-M.-Pinsker).
- fo-definitions in $(\mathbb{Z}; \mathit{succ})$ (Bodirsky-M.-Mottet).

Classifications for infinite CSPs

ω-categorical and/ or using algebraic method?

- $\surd$ Allen's Interval Algebra (Jeavons-Jonsson-Krokhin).
- $\surd$ fo-definitions in $(\mathbb{Q}; <)$ (Bodirsky-Kára).
- $\surd$ fo-definitions in $\mathcal{RG}$ (Bodirsky-Pinsker).
- $\times$ fo-expansions of $(\mathcal{R}; +, 1, \leq)$ (Bodirsky-Jonsson-von Oertzen).
- $\times$ fo-expansions of $(\mathcal{R}; +, 1)$ using $\leq$ (Jonsson-Thapper).
- ? bounded Gaifman-degree fo-definitions in $(\mathbb{Z}; succ)$ (Bodirsky-Dalmau-M.-Pinsker).
- ? fo-definitions in $(\mathbb{Z}; succ)$ (Bodirsky-M.-Mottet).

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Theorem (Bodirsky, Hils and M. 2010)

Let $\mathcal{B}$ be a saturated structure of cardinality $\geq 2^\omega$. Then

$$\mathsf{Inv}(\mathsf{Pol}^\omega(\mathcal{B})) \cap \langle \mathcal{B} \rangle_{\mathrm{fo}} = \langle \mathcal{B} \rangle_{\mathrm{pp}}.$$

Furthermore, we can show that each of

- $\mathcal{B}$ being big and saturated,
- using Pol^ω instead of Pol, and
- taking intersection with $\langle \mathcal{B} \rangle_{\mathrm{fo}}$

is necessary.

There is no simple characterisation to the Galois converse here.

Distance CSPs have a template fo-definable in $(\mathbb{Z}; succ)$.

- We should have called these Successor CSPs!

Theorem (Bodirsky-Dalmau-M.-Pinsker 2010)

Let $\mathcal{B}$ be fo-definable $(\mathbb{Z}; succ)$ with finite-degree Gaifman graph. Then either

- *$\mathcal{B}$ is homomorphically equivalent to a finite transitive core, or*
- *$\mathcal{B}$ has a modular median poly and CSP$(\mathcal{B})$ is in P, or*
- *CSP$(\mathcal{B})$ is NP-complete.*

Although this result uses endos, it is combinatorial, not algebraic, in flavour.

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$(\mathbb{Z}; succ)$ is not ω-categorical, but it is 2^ω-categorical.

- Its big models are simple!

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

becomes

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\vdots \;\cdots\; \vdots \;\cdots\; \vdots$$

and it admits quantifier elimination in its functional form. E.g.

$$\exists z \; y = succ(z) \wedge z = succ(x)$$

becomes $y = succ^2(x)$.

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Finite signature and finite-degree Gaifman mean finite

distance-degree $:= \max\{|x - y| : x, y$ appear in a relation tuple$\}$

For example,

- $\surd$ $y = succ(x) \vee y = succ^2(x)$.
- $\times$ $\neq$.
- $\times$ $y = succ(w) \vee y = succ(x)$.

Distance CSPs with bounded distance-degree represent a tiny subclass of distance CSPs in general.

But bounded distance-degree is so useful!

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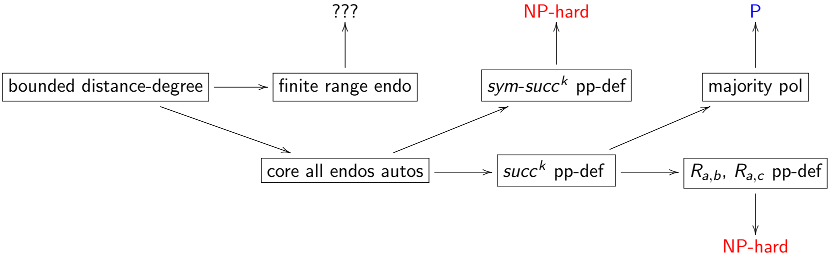
Outline bounded distance-degree case



This provides a first step to the general case.

Recette

How to get a handle on the general case???

- We already met another type of degree!

For R fo-definable in $(\mathbb{Z}; succ)$, the qe-degree is the minimal nesting of functional *succ* in its qf-definition.

Key ingredients:
old classification
bounded qe-degree
the ω-saturated model
Bodirsky et al. tricks

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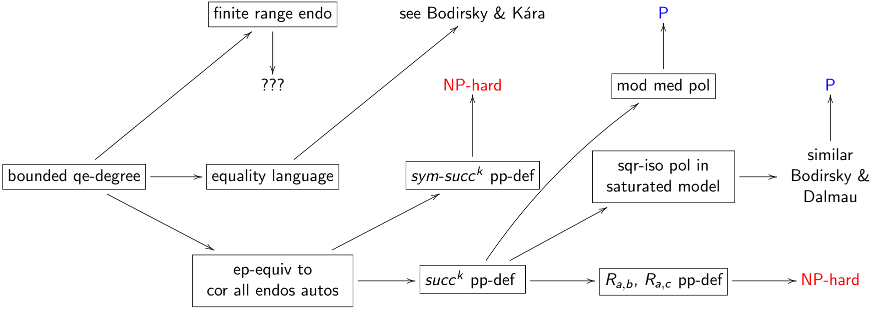
new classification!

Outline general case



Theorem (Petrus)

Let Γ be a reduct of $(\mathbb{Z}; succ)$ with fin sig and no endo of finite range. TFAE:

- *there exists a reduct Δ of $(\mathbb{Z}; =)$ such that $\mathrm{CSP}(\Delta)$ equals $\mathrm{CSP}(\Gamma)$;*
- *$\omega.\Gamma$ has an endo whose range induces a struct iso to a reduct of $(\mathbb{Z}; =)$;*
- *for all $t \geq 1$, there is an $e \in \mathrm{End}(\Gamma)$, $z \in \mathbb{Z}$, so that $|e(z+t) - e(z)| > t$;*
- *all binary $R \in \langle \Gamma \rangle_{\mathrm{pp}}$ are either $=$ or have unbounded dist degree;*
- *there exists an $e \in \mathrm{End}(\omega.\Gamma)$ with inf range s.t. $e(x) - e(y) = \omega$ or $e(x) = e(y)$ for any two distinct $e(x), e(y) \in \omega.\Gamma$.*

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Hauptsatz

Theorem (Bodirsky-M.-Mottet 2013)

Let $\mathcal{B}$ be fo-definable $(\mathbb{Z}; succ)$ with finite signature. Then either

- $\mathcal{B}$ *is hom equivalent to a finite transitive core, or*
- $\mathcal{B}$ *is hom equivalent to an equality language, or*
- *equiv* $\mathcal{B}$ *has a modular median poly and CSP*$(\mathcal{B})$ *is in P, or*
- *equiv* $\omega\mathcal{B}$ *has special binary poly and CSP*$(\mathcal{B})$ *is in P, or*
- *CSP*$(\mathcal{B})$ *is NP-complete.*

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We need finite signature for finite distance-degree but we even say, for relations coded in DNF:

Theorem (Bodirsky-M.-Mottet 2013)

Let $\mathcal{B}$ be fo-definable $(\mathbb{Z}; succ)$. Then either

- *$\mathcal{B}$ is hom equivalent to a finite transitive core, or*
- *$\mathcal{B}$ is hom equivalent to an equality language, or*
- *equiv $\mathcal{B}$ has a modular median poly and CSP($\mathcal{B}$) is in P, or*
- *equiv $\omega\mathcal{B}$ has special binary poly and CSP($\mathcal{B}$) is in P, or*
- *CSP($\mathcal{B}$) is NP-complete.*

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Open questions

Fo-definitions in $(\mathbb{Z}; succ, 0)$ embed all finite CSPs.

- Does fo-definitions in $(\mathbb{Z}; succ, 0)$ have a non-dichotomy?
  - finite signature?
  - infinite signature, relations in DNF?
- fo-definitions in $(\mathbb{Z}; \leq, succ)$?
  - finite signature?
  - infinite signature, relations in DNF?
- fo-definitions in $(\mathbb{Z}; \leq, +, 0)$?

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Open questions

Fo-definitions in $(\mathbb{Z}; \mathit{succ}, 0)$ embed all finite CSPs.

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  - finite signature?
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- fo-definitions in $(\mathbb{Z}; \leq, \mathit{succ})$?
  - finite signature?
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- fo-definitions in $(\mathbb{Z}; \leq, +, 0)$?

** CSP$(\mathbb{Z}; \mathit{succ}, \mathit{succ}^2, \ldots, x \geq y \vee z \geq y)$ is MAX ATOMS. **

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