Restricted Dualities and First-Order Definable Colorings

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Outline

Problem

Given a fixed class $\mathcal{C}$ of relational structures, determine which H-coloring problems are first-order definable in $\mathcal{C}$.

$$\exists \phi_H \quad \forall G \in \mathcal{C} : \quad (G \models \phi_H) \iff (G \to H)$$

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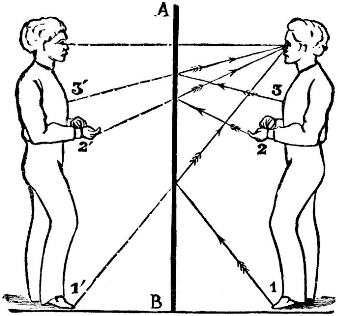
$$\exists \phi_H \quad \forall G \in \mathcal{C} : \quad (G \models \phi_H) \iff (G \to H)$$

- $\mathcal{C}$ = all finite structures;
- smaller $\mathcal{C}$;
- bigger $\mathcal{C}$.

First-Order definable coloring and Duality



Finite Structures

Theorem (Atserias '05; Rossman '08)

Every H-coloring which is first-order definable on finite structures corresponds to a finite duality.

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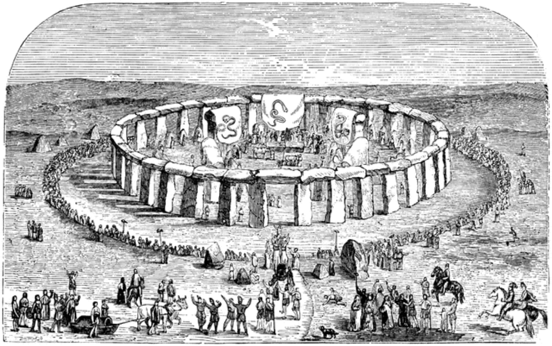
Theorem (Larose, Loten,Tardif '07)

H-coloring is first-order definable if and only if H has a retract whose square dismantles to its diagonal.

(Deciding whether H-coloring is first-order definable is NP-complete)

Restricting the Class



Smaller Classes (Trivial Case)

Remark

If a class $\mathcal{C}$ is well quasi-ordered by homomorphism order then every H-coloring problem on $\mathcal{C}$ is first-order definable.

Example

On the class of m-partite cographs, every hereditary property (for instance existence of a H-coloring) is first-order definable and can be tested in linear time.

Smaller Classes (Monotone Case)

Definition

Let $\mathcal{C}$ be a class of graphs.

- For $p \in \mathbb{N}$, $\mathcal{C} \,\widetilde{\nabla}\, p$ is the class of all depth p topological minors of $G \in \mathcal{C}$;
- $\mathcal{C}$ is nowhere dense if

$$\forall p \quad \sup\{\,\omega(H) :\ H \in G \,\widetilde{\nabla}\, p\} < \infty;$$

$\mathcal{C}$ has bounded expansion if

$$\forall p \quad \sup\{\,\overline{\mathrm{d}}(H) :\ H \in G \,\widetilde{\nabla}\, p\} < \infty;$$

Nowhere Dense Classes

Theorem (Adler, Adler '14)

Let $\mathcal{C}$ be a monotone class of coloured digraphs of a fixed finite signature. The following conditions are equivalent.

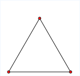
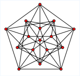
1. $\underline{\mathcal{C}}$ is nowhere dense;
2. $\underline{\mathcal{C}}$ is superflat.
3. $\mathcal{C}$ is stable.
4. $\underline{\mathcal{C}}$ is stable.
5. $\mathcal{C}$ is dependent.
6. $\underline{\mathcal{C}}$ is dependent.

Restricted dualities

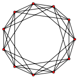
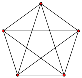
$\mathcal{C}$-restricted duality

$$\begin{cases} \forall G \in \mathcal{C}: & (\forall F \in \mathcal{F},\ F \not\longrightarrow G) \iff (G \longrightarrow D) \\ \forall F \in \mathcal{F}: & F \not\longrightarrow D. \end{cases}$$

Example 1 (Naserasr): $\forall$ planar G

 $\not\longrightarrow G \iff G \longrightarrow$ 

Example 2 (Thomassen): $\forall$ toroidal G

 $\not\longrightarrow G \iff G \longrightarrow$ 

Classes with all restricted dualities

Definition

A class $\mathcal{C}$ has *all restricted dualities* if every connected F has a dual D for $\mathcal{C}$: $F \nrightarrow D$ and

$$\forall G \in \mathcal{C}, \qquad (F \nrightarrow G) \iff (G \rightarrow D).$$

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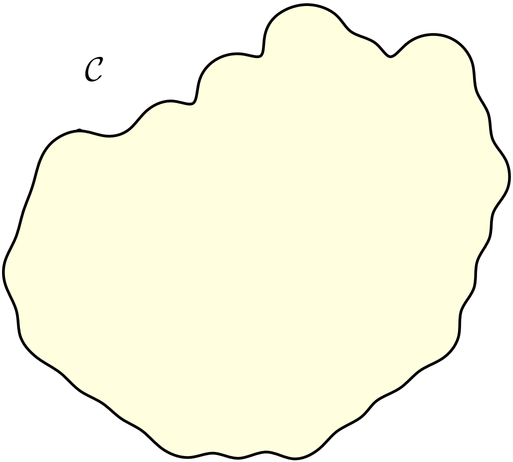
Theorem (Nešetřil, POM)

A bounded class $\mathcal{C}$ has all restricted dualities if and only if $\forall p \in \mathbb{N} \quad \forall G \in \mathcal{C} \quad \exists \Phi_p(G)$ such that:

1. $|\Phi_p(G)| \leq F(\mathcal{C}, p)$;
2. $G \to \Phi_p(G)$;
3. $\forall F, |F| \leq p : \quad (F \to G) \iff (F \to \Phi_p(G))$.

$\Phi_p \implies$ restricted dualities



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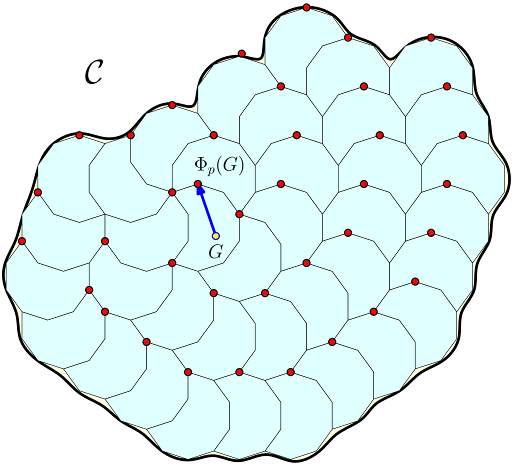

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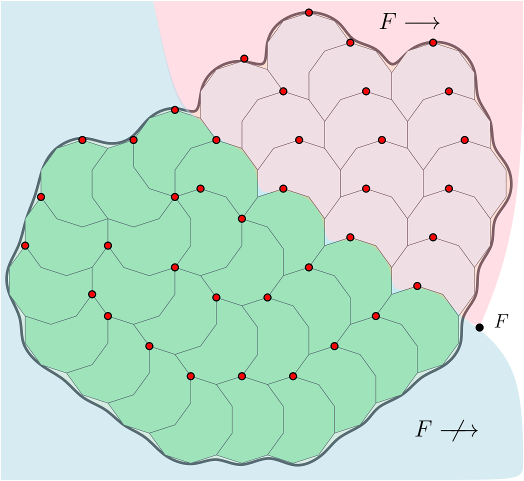

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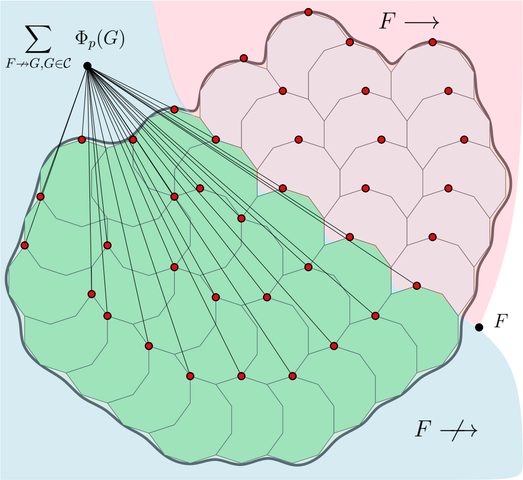

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Theorem (Nešetřil, POM)

Let $\mathcal{C}$ be a class of directed graphs closed under reorientations.
$\mathcal{C}$ has all restricted dualities $\iff$ $\mathcal{C}$ has bounded expansion.

Let $\mathcal{C}$ be a class of undirected graphs closed under subdivisions.
$\mathcal{C}$ has all restricted dualities $\iff$ $\mathcal{C}$ has bounded expansion.

First order definable H-coloring

H-coloring is *first-order definable* on $\mathcal{C}$ if there exists a first-order formula φ_H such that

$$\forall G \in \mathcal{C} \qquad (G \models \varphi_H) \iff (G \to H).$$

FO–BE Conjecture

Let $\mathcal{C}$ be a hereditary addable class of graphs closed by subdivisions. The following are equivalent:

- there exists in $\mathcal{C}$ first-order definable H-colorings for non bipartite H of arbitrarily large odd-girth;
- the class $\mathcal{C}$ has bounded expansion.

Reduction Step

- Reduction from first-order definable H-coloring to restricted duality by Rossman '08 or Dawar '10;
- Idea: if $\mathcal{C}$ has not bounded expansion, for H with sufficiently large girth, find $G \in \mathcal{C}$ such that

$$(\forall F \in \mathcal{F})\ F \nrightarrow G \quad \text{and} \quad G \nrightarrow H.$$

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Conjecture

Let $\mathcal{C}$ be a monotone nowhere dense class that does not have bounded expansion. Then there exists an integer p such that $\mathcal{C}$ includes p-subdivisions of graphs with arbitrarily large chromatic number and girth.

Two ways to the FO–BE conjecture. . .

Would follow (in a non-trivial way) from any of the two following conjectures:

Erdős–Hajnal

$\forall c, g, \ \exists f(c,g)$ s.t.

$\chi(G) \geq f(c,g) \ \Rightarrow \ \exists H \subseteq G$

- $\chi(H) \geq c$;
- $\mathrm{girth}(H) \geq g$.

Thomassen

$\forall c, g, \ \exists f(c,g)$ s.t.

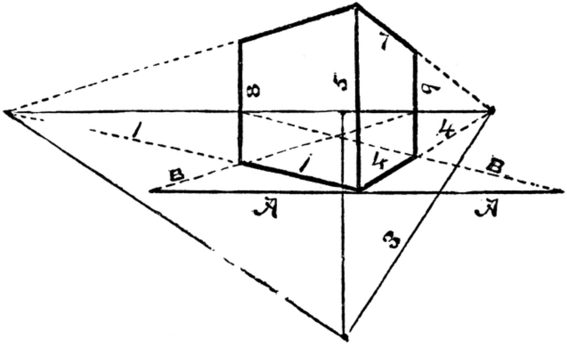
$\overline{\mathrm{d}}(G) \geq f(c,g) \ \Rightarrow \ \exists H \subseteq G$

- $\overline{\mathrm{d}}(H) \geq c$;
- $\mathrm{girth}(H) \geq g$.

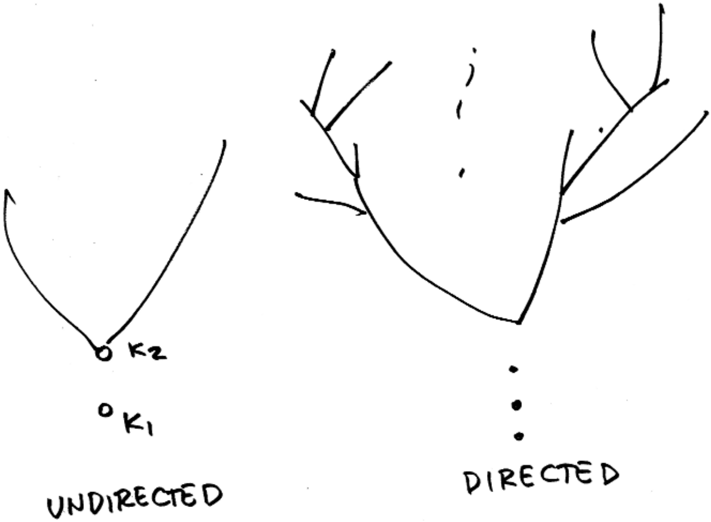
$g = 4$: Rődl '77

$g = 6$: Kuhn and Osthus '04

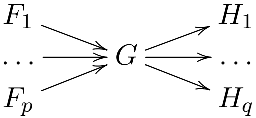
Extending the Class



Homomorphism Order



Homomorphism Order and Distance



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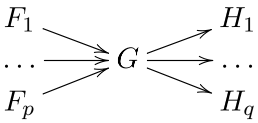

$$(\to G) = \{F : \ F \to G\}$$
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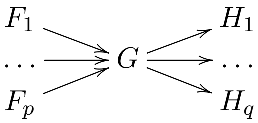

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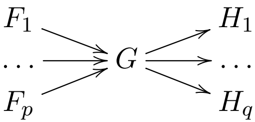

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$$\mathrm{dist}(A, B) = 2^{-\min\{|F| : \; F \in (\to A)\,\Delta\,(\to B) \;\cup\; (A \to)\,\Delta\,(B \to)\}}$$

Limits

Consider the completion of the homomorphism order.

Theorem (Nešetřil, POM)

Objects $\mathbb{L}$ are the pairs (A, B), where

- A is an ideal;
- B is a filter;
- $\forall (G, H) \in A \times B$, it holds $G \to H$;
- if T is a tree and $T \to H$ holds for all $H \in B$ then $T \in A$.

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We note $(\to \mathbb{L}) = A$ and $(\mathbb{L} \to) = B$.

Full Duality

$$\mathbb{A} \leq \mathbb{B} \iff (\to \mathbb{A}) \subseteq (\to \mathbb{B}) \text{ and } (\mathbb{A} \to) \supseteq (\mathbb{B} \to).$$

Definition

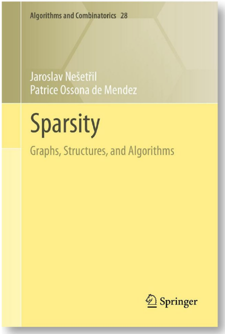
A full duality is a pair $(\mathbb{F}, \mathbf{D})$ such that $\forall \mathbb{L}$ it holds

$$\mathbb{F} \not\leq \mathbb{L} \iff \mathbb{L} \leq \mathbb{D},$$

Theorem (Nešetřil, POM)

Every connected finite structure has a full dual,
every multiplicative finite structure is a full dual,
and these are the only full dualities.

Conclusion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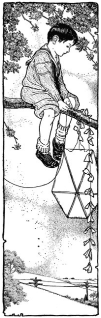

Thank you for your attention.