Continuity of homomorphisms to the clone of projections

András Pongrácz

School of Science and Technology, Middlesex University

joint work with Manuel Bodirsky and Michael Pinsker

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item (1) $\Rightarrow$ $\mathrm{CSP}(\Delta)$ is NP-complete (WELL-KNOWN)

item (2) $\Rightarrow$ $\mathrm{CSP}(\Delta) \in \mathrm{P}$

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- countable vector spaces over $GF(q)$, countable atomless Boolean algebra

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Question 2: What can we say in the two cases about the complexity?

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Bodirsky, Pinsker

TFAE for an ω-categorical Δ.

- $\exists \Phi : \mathrm{Pol}(\Delta) \to \mathbf{1}$ continuous homomorphism.
- All finite structures have a primitive positive interpretation in Δ (and in particular, CSP(Δ) is NP-hard).

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Remark. In that case, there exist continuous ones, too.

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A unary function $f : \Delta \to \Delta$ is *canonical* if whenever $\underline{a}, \underline{b} \in \Delta^k$ have the same type, then $f(\underline{a})$ and $f(\underline{b})$ have the same type (for all k).

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Proof. The Φ^{typ}-image of an n-ary f depends only on the restriction of f to a big enough finite set.

Bodirsky, Pinsker, AP ’14

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Theorem. Let Δ be a homogeneous structure in a finite relational language, and let $\mathcal{C}$ be a closed canonical clone. If $\exists \Phi : \mathcal{C} \to \mathbf{1}$ homomorphism, then there is also a continuous $\mathcal{C} \to \mathbf{1}$ homomorphism.

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Corollary 1. The clone dichotomy holds for closed canonical clones.

Corollary 2. Let Δ be homogeneous in a finite relational language. Assume that $\mathcal{C} = \text{Pol}(\Delta)$ is a closed canonical clone, and that $\exists \Phi : \mathcal{C} \to \mathbf{1}$ homomorphism. Then CSP(Δ) is NP-hard.

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Moreover, $\text{Pol}(\Delta) = \overline{\bigcup_{i=1}^{\infty} \mathcal{C}_i}$, where each $\mathcal{C}_i$ is a closed canonical clone.

Temporal constraints and GRAPH-SAT

Bodirsky, Kára

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Bodirsky, Pinsker

Let Δ be first-order definable in the random graph. Then CSP(Δ) is either in P or NP $-$*complete*.

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- Let $\mathfrak{A}$ be an algebra with a countable base set. Is it true that whenever $\mathbf{2} \in \mathrm{HSP}(\mathfrak{A})$, then $\mathbf{2} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A})$?

Some open problems

- Is there a closed clone $\mathcal{C}$ with a homomorphism to $\mathbf{1}$ but no continuous one?
- Let $\mathfrak{A}$ be an algebra with a countable base set. Is it true that whenever $\mathbf{2} \in \mathrm{HSP}(\mathfrak{A})$, then $\mathbf{2} \in \mathrm{HSP}^{\mathrm{fin}}(\mathfrak{A})$?
- Is there a model of ZF in which every homomorphism from a closed clone to $\mathbf{1}$ is continuous?