Endomorphisms and synchronization, 2: Graphs and transformation monoids

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What about going in the other direction?

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First a very brief introduction to graph homomorphisms.

Homomorphisms of graphs

A homomorphism from graph Γ_1 to Γ_2 is a map from vertices of Γ_1 to vertices of Γ_2 which maps edges to edges. Its action on non-edges is unrestricted: a non-edge may map to a non-edge, or to an edge, or may collapse to a single vertex.

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- *There is a homomorphism from Γ to K_r if and only if $\chi(\Gamma) \leq r$, where χ denotes the chromatic number.*

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The following are equivalent:

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- $\omega(\Gamma) = \chi(\Gamma) = r$.

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If $M_1 \leq M_2$*, then* $\mathrm{Gr}(M_2)$ *is a spanning subgraph of* $\mathrm{Gr}(M_1)$*.*

(A spanning subgraph of Γ uses all the vertices and some of the edges of Γ.)

Going both ways

Proposition

For any transformation monoid M, we have $M \leq \mathrm{End}(\mathrm{Gr}(M))$.

Proof.

Let $\{v, w\}$ be an edge of $\mathrm{Gr}(M)$, and take $f \in M$. We must show that f maps $\{v, w\}$ to an edge of $\mathrm{Gr}(M)$. If not, there are two possibilities:

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- $vf = wf$: this contradicts the fact that $\{v, w\}$ is an edge of $\mathrm{Gr}(M)$.
- $\{vf, wf\}$ is a non-edge of $\mathrm{Gr}(M)$: then by definition, there exists $g \in M$ such that $(vf)g = (wf)g$. So $v(fg) = w(fg)$ with $fg \in M$, again contradicting the fact that $\{v, w\}$ is an edge of $\mathrm{Gr}(M)$.

$\square$

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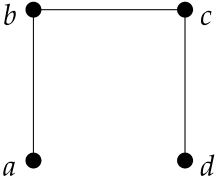

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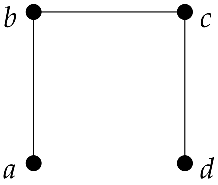

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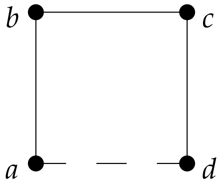

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So both the operators $M \mapsto \mathrm{End}(\mathrm{Gr}(M))$ on monoids and $\Gamma \mapsto \mathrm{Hull}(\Gamma)$ on graphs are idempotent.

Cores and hulls

The hull of a graph is, in some sense, a “dual” to the core. Core(Γ) is an induced subgraph of Γ (the smallest graph hom-equivalent to Γ); Hull(Γ) is a graph containing Γ as a spanning subgraph.

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- … but the converse of this statement is false.
- In particular, $\mathrm{Core}(\mathrm{Hull}(\Gamma))$ is a complete graph on $\mathrm{Core}(\Gamma)$.

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Corollary

- $\mathrm{Gr}(M)$ *is complete if and only if M is a permutation group (i.e., contained in the symmetric group).*
- $\mathrm{Gr}(M)$ *is null if and only if M is synchronizing (i.e. contains a transformation of rank 1).*

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$\mathrm{Gr}(M)$ has the second and third properties of the theorem, and is non-null if and only if M is not synchronizing.
Conversely, if M is contained in the endomorphism monoid of a non-null graph, then no edge of the graph is collapsed by M, and so M is not synchronizing. □

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This extra property is sometimes useful. An application follows.

Maximal non-synchronizing monoids

Theorem

Let M be a transformation monoid on n points which is maximal with respect to being non-synchronizing. Then there are graphs Γ and Δ such that

- $\mathrm{End}(\Gamma) = \mathrm{End}(\Delta) = M$*;*
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- $\Gamma = \mathrm{Hull}(\Delta)$ *and* $\Delta = \Gamma'$*.*

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Problem

Find a necessary and sufficient condition!

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Note that the theory of finite permutation groups, the oldest part of group theory, has been revolutionised by the classification of finite simple groups. I will have more to say about this later.

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If you know other definitions of these concepts you should have no trouble matching them up with the ones given here.

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Proof.

$\Leftarrow$: If G preserves a non-trivial equivalence relation $\equiv$, and $v \equiv w$, then the map sending v to w and fixing all other points is not synchronized by G. $\square$

Proof.

$\Rightarrow$: Conversely, suppose that $M = \langle G, f \rangle$ is not synchronizing, where f has rank $n-1$, and let $\Gamma = \mathrm{Gr}(M)$. Suppose that v and w have the same image under f. Then v and w are not joined in Γ. So the neighbours of v and of w are both mapped bijectively to the neighbours of $vf = wf$ by f, and thus these neighbour sets are equal. Putting $x \equiv y$ if x and y have the same neighbour sets, we obtain a non-trivial G-invariant equivalence relation, so G is imprimitive. □

A note on primitivity

We saw that primitivity, as I defined it earlier, is exactly equivalent to synchronizing every map of rank $n - 1$.

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However, there is one small awkwardness. According to this definition, the trivial group acting on a set of cardinality 2 is primitive, even though it is not transitive!
It is usual to exclude this exceptional case, and to re-define primitivity in such a way that a primitive group is transitive. If this is done (as I shall assume in future), then we must add to Rystsov's Theorem the assumption that $n > 2$.

Primitivity and synchronization

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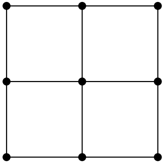
The answer is no. The picture shows the 3×3 grid, whose automorphism group is the primitive group $G = S_3 \wr S_2$. (Vertices in the same row or the same column are joined.)



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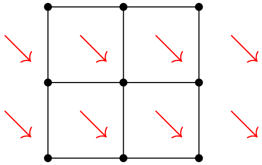
The answer is no. The picture shows the 3×3 grid, whose automorphism group is the primitive group $G = S_3 \wr S_2$. (Vertices in the same row or the same column are joined.) The map taking each vertex to the vertex in the bottom row obtained by moving south-east (wrapping round if necessary) is a graph endomorphism and so is not synchronized by G.



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- Compute all the non-trivial G-invariant graphs.
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- If the answer is ever "yes", then G is non-synchronizing; otherwise it is synchronizing.

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However, the algorithm is often better than it seems, and has been used to test the synchronizing property for groups with degrees up to several thousand.

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2. Although both problems are NP-hard, in practice the clique number is much easier to compute than the chromatic number (and indeed parametrised complexity theory gives an explanation of this), and often synchronization can be proved with just clique nunber calculations.

Non-basic groups

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The k-dimensional cube graph over an alphabet of size m (with $n = m^k$ vertices) has endomorphisms onto the l-dimensional subcube for $1 \leq l \leq k$, with image of size m^l and kernel classes of size m^{k-l}.

Observing that the example on the last slide is non-basic, we might wonder whether basic primitive groups are necessarily synchronizing.

Basic groups

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- The line graph of the complete graph K_m has clique number $m - 1$, and chromatic number $m - 1$ if m is even. Its automorphism group is the symmetric group S_n (acting on 2-sets).
- A classical polar space defines two graphs, where the adjacency relation is orthogonality or non-orthogonality respectively. The first has $\omega = \chi$ if and only if the polar space has a partition into ovoids, and the second if and only if the polar space has both an ovoid and a spread. The automorphism group is the corresponding classical group.

The O'Nan–Scott Theorem

One part of the O'Nan–Scott Theorem says that basic primitive groups are of three types:

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Examples of other types also occur.
So far, we are some way from a classification of the synchronizing groups which are basic primitive groups.

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Let A be a maximum clique, and B a colour class in a minimum colouring. Then $Ag \cap B \neq \emptyset$ for all automorphisms g. But this inequality for all elements of a transitive group implies that $|A| \cdot |B| = n$, the number of vertices. So $|B|$ is independent of the chosen colour class. □

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Many other examples of uniform endomorphisms can be constructed, such as the endomorphisms of the k-dimensional cube graph mentioned earlier.

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Various special cases of the conjecture were proved. However, last month the first counterexample appeared, and now we have an extremely interesting list of non-uniform maps synchronized by primitive groups. We'll see them in the final lecture . . .