Linear Datalog and k-permutability = symmetric Datalog

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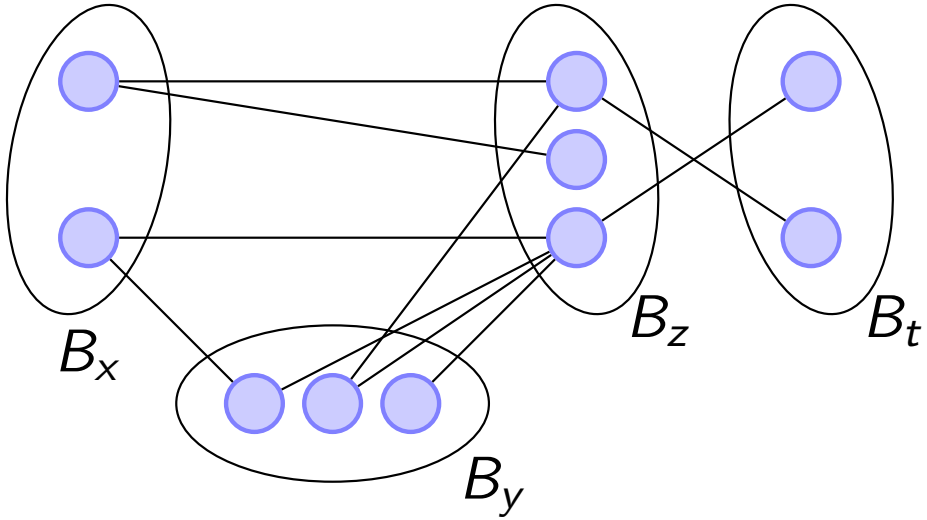
Constraint Satisfaction Problem



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- Is there a solution?

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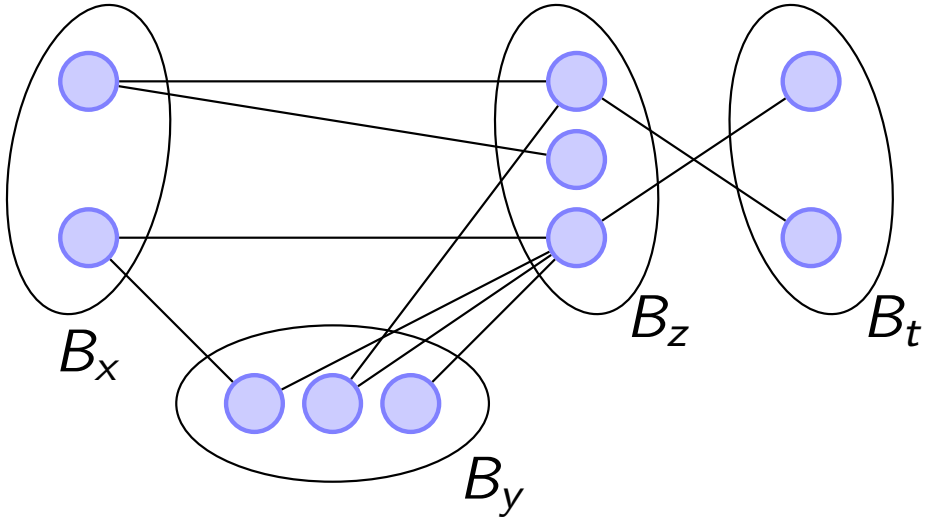

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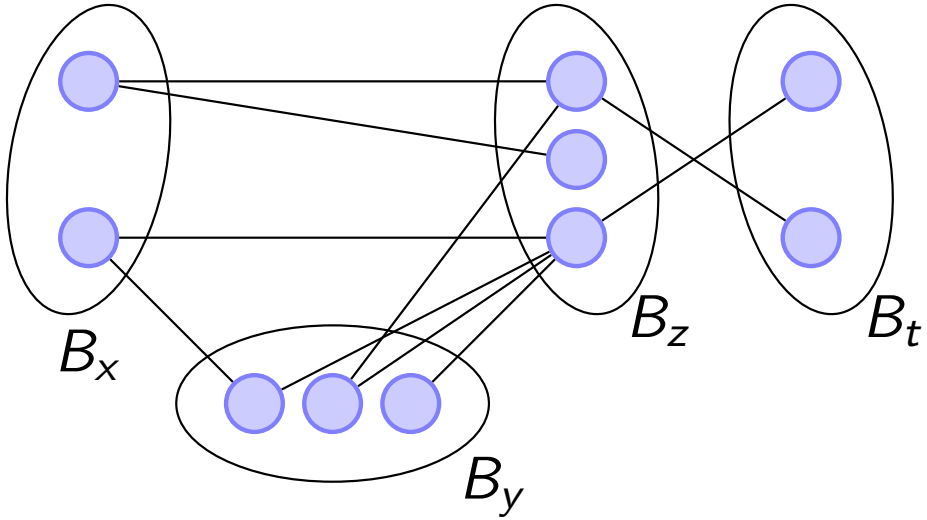

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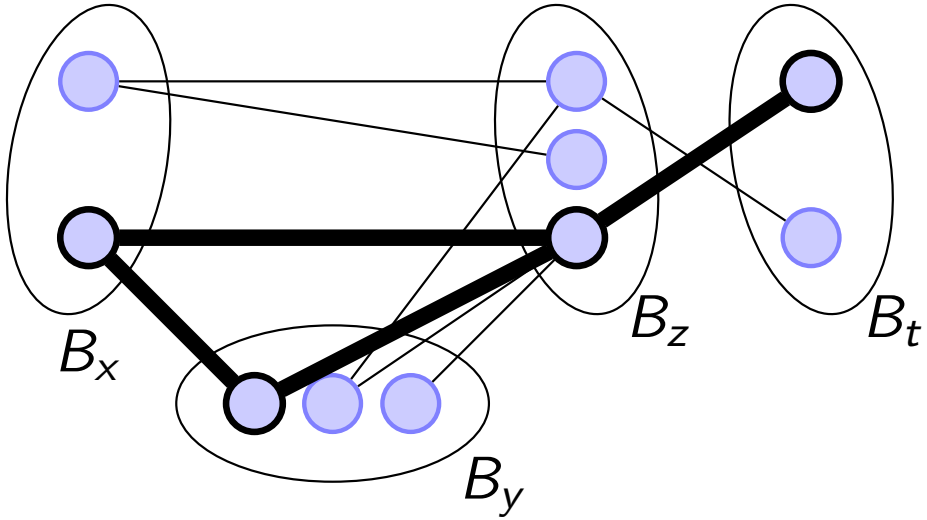

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Consistency checking

- Solving CSP is NP-complete in general.
- If a part of a CSP instance has no solution, then the whole instance has no solution.

- Good news: Consistency can be checked quickly.
- Bad news: An instance can have no solution, yet be locally consistent.

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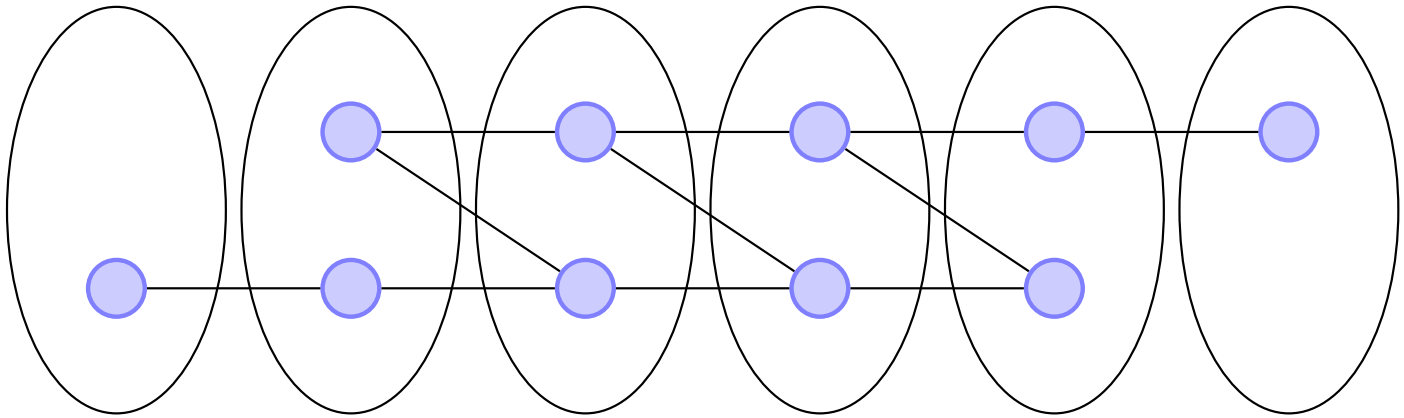

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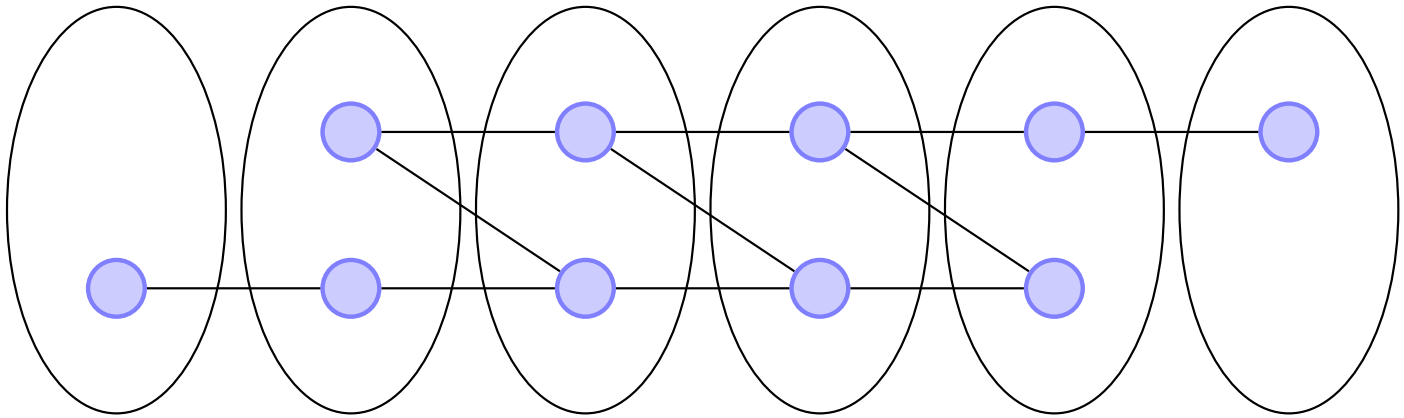

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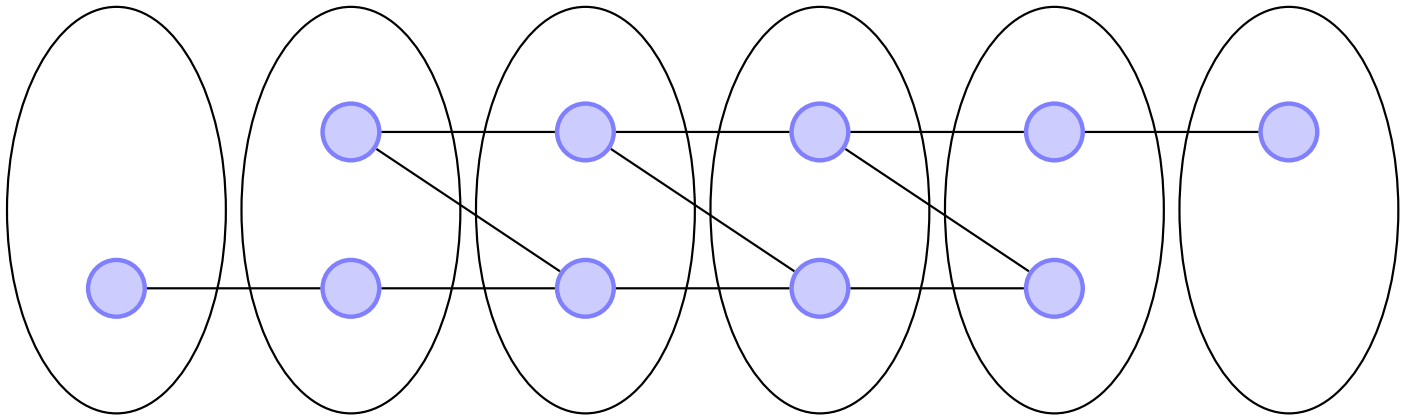

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Datalog

- A Datalog program adds tuples to predicates using local rules until it reaches its goal or can't apply any rule.
- Example of a Datalog program with unary predicates Z, c_0, c_1, binary predicate E, and the goal predicate G.

$$Z(x) \Leftarrow c_0(x)$$
$$Z(y) \Leftarrow Z(x) \wedge E(x, y)$$
$$G \Leftarrow Z(y) \wedge c_1(x)$$

- The above program checks if there is a directed path from something in c_0 to something in c_1.
- How to use Datalog to solve $\neg$ CSP: A Datalog program verifies local consistency. Goal: Prove inconsistency.

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- Symmetric Datalog: At most one IDB on the right hand side of any rule. If there is an IDB on both sides of a rule, we can switch the IDBs.
- Example: Rule

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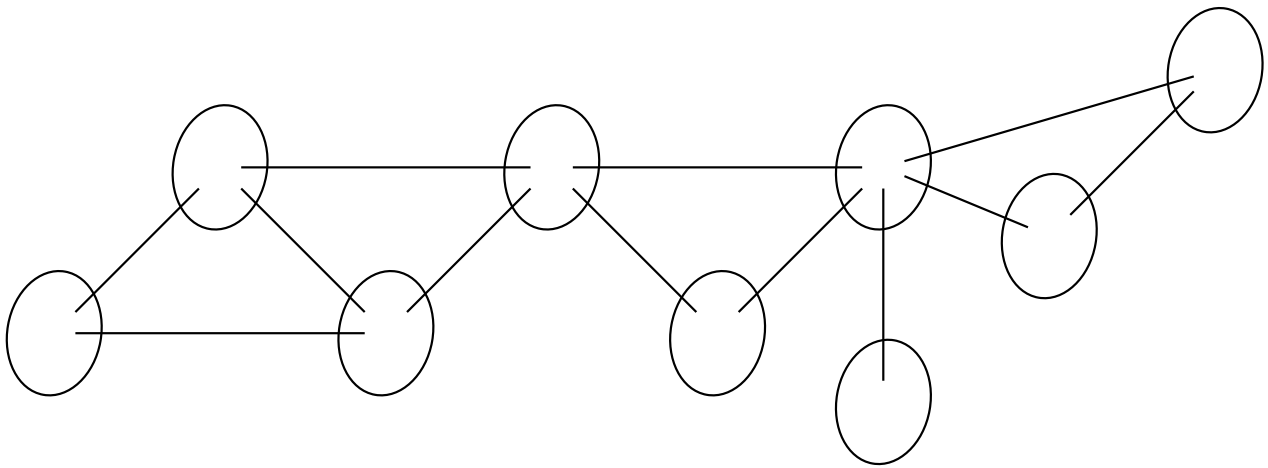

Solving path instances is enough

Theorem

If $\neg\,\mathrm{CSP}(\mathbb{A})$ *is solvable by a linear Datalog program and* $\mathbb{A}$ *is* n*-permutable for some* n*, then* $\neg\,\mathrm{CSP}(\mathbb{A})$ *can be solved by a symmetric Datalog program.*

- $\mathbb{A}$ has bounded pathwidth duality.
- It is enough to look at path shaped unsatisfiable CSP instances.

- How to use symmetric Datalog to decide path CSP instances?

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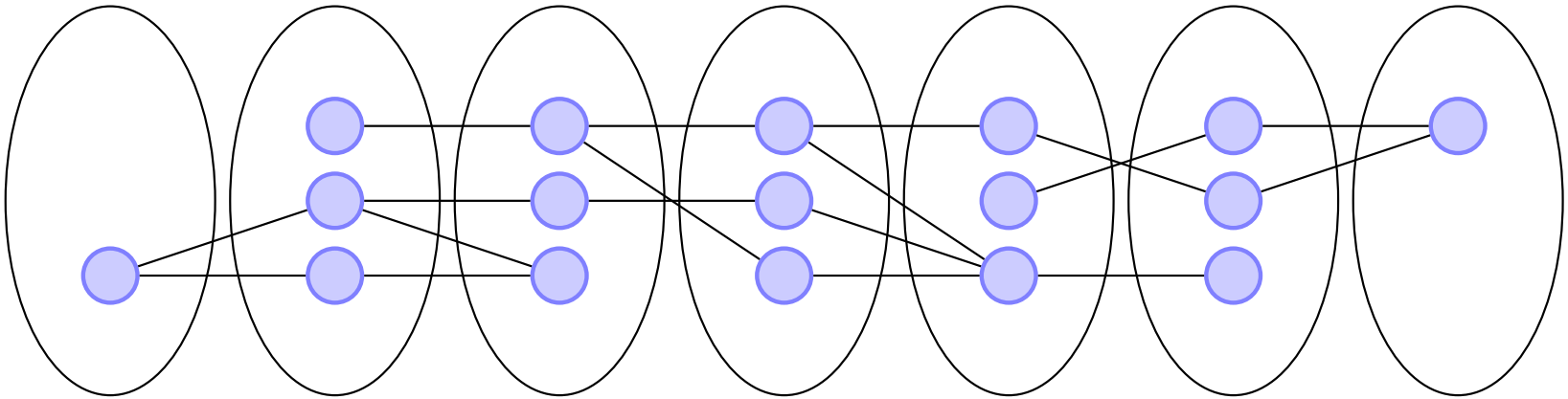

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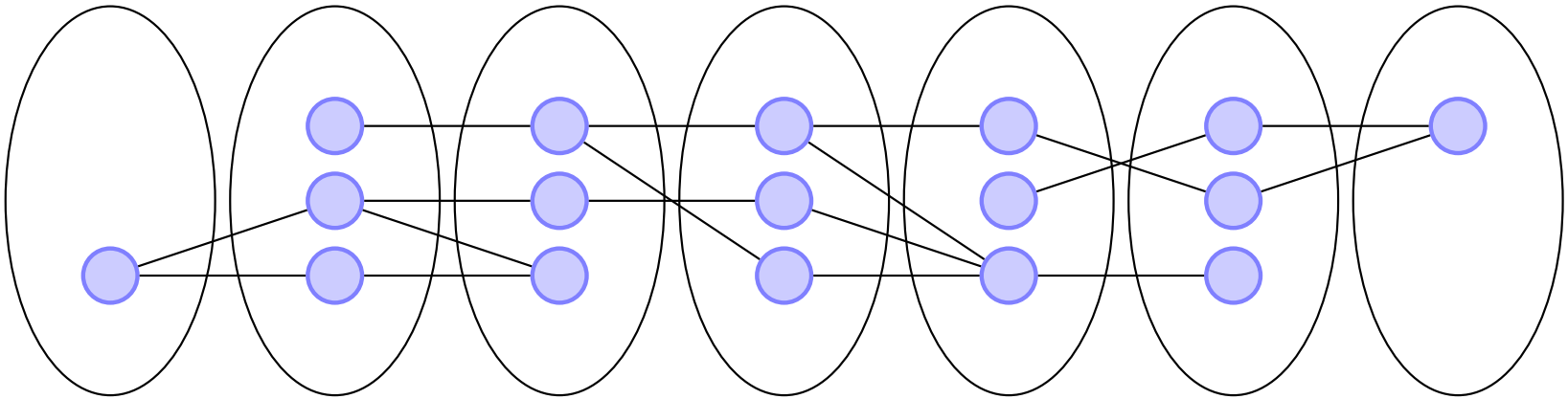

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Using n-permutability

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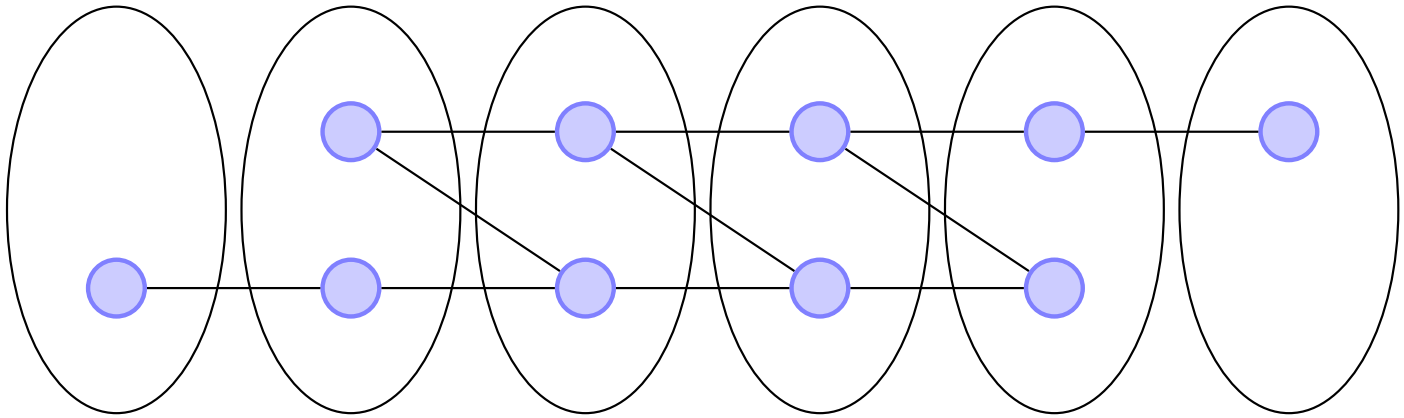

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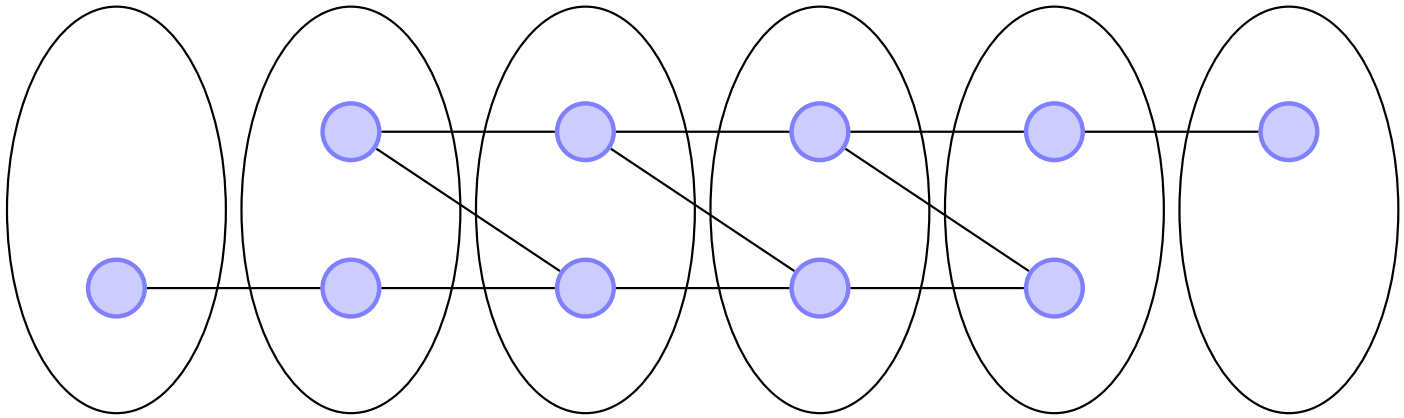

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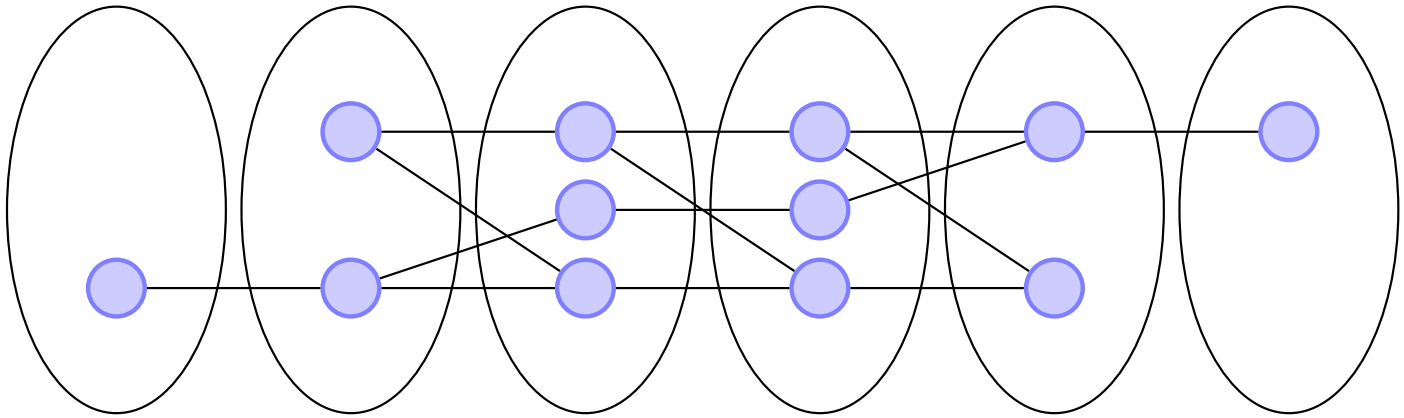

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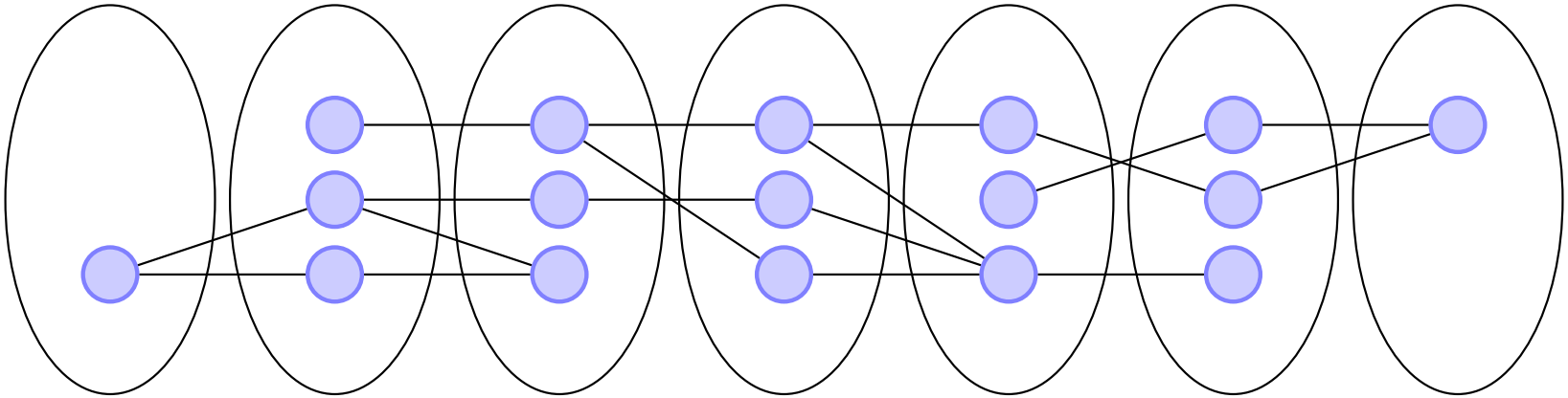

- We label black the vertices that can be reached from the starting potato.
- We want lots of backwards edges: Edges from blue to black vertices.
- We use some trickery to glue together potatoes without blue to black edges.
- Nature of the trickery: We can run a smaller symmetric Datalog program inside our original program.

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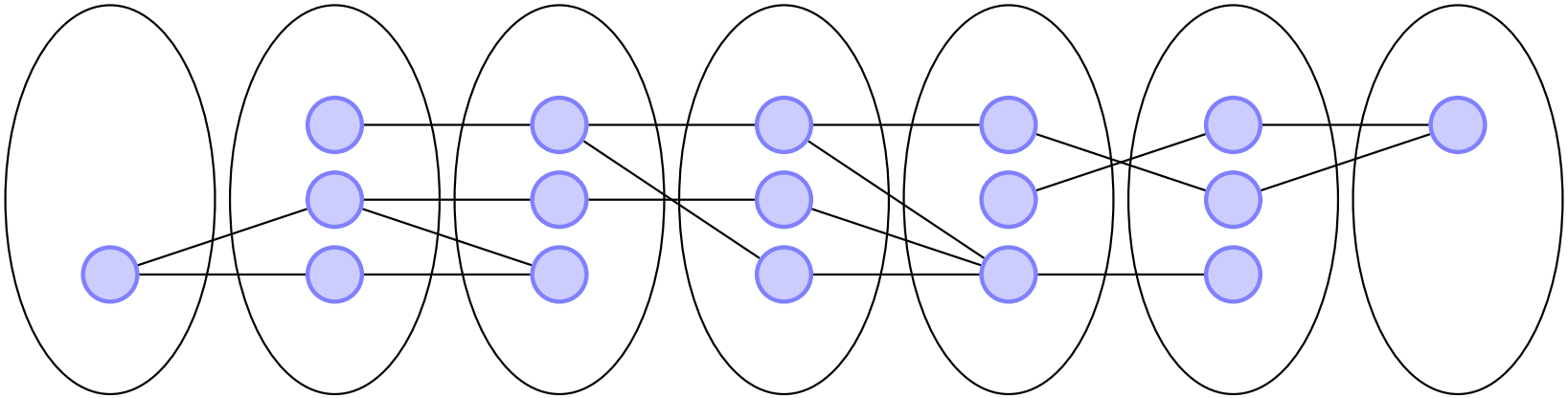

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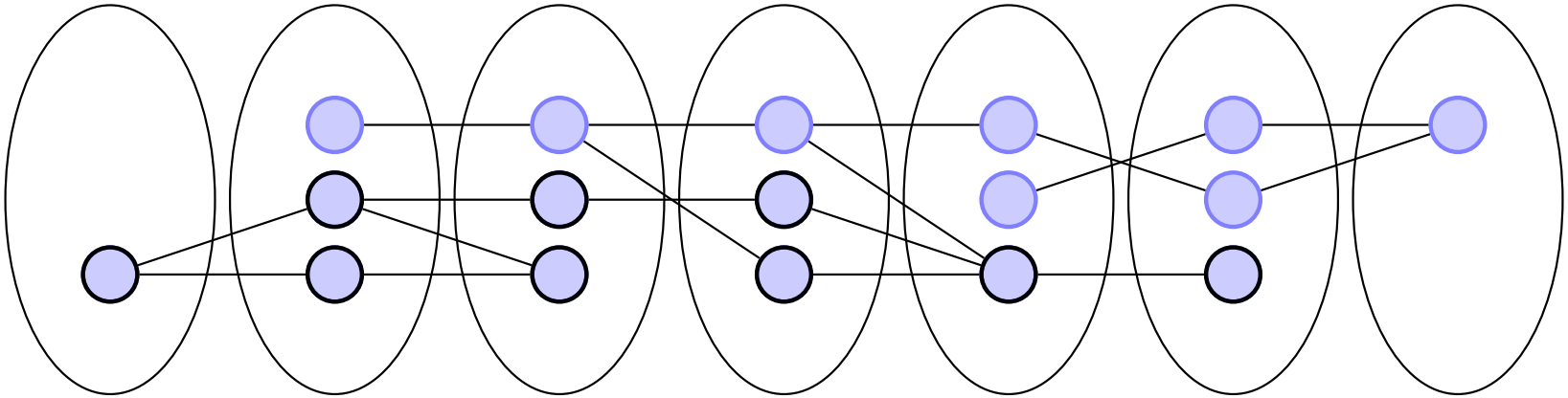

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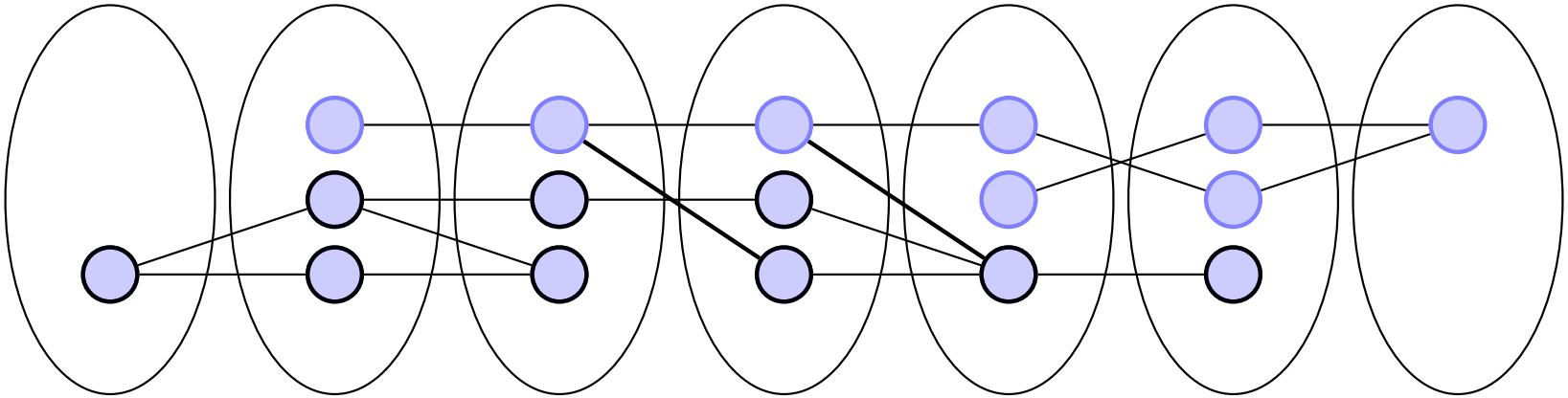

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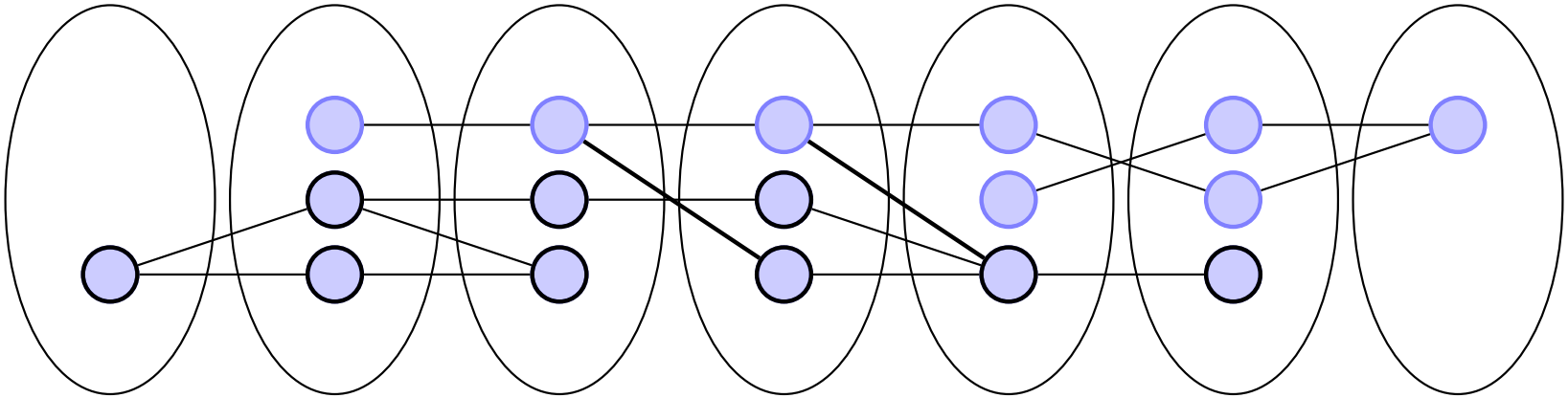

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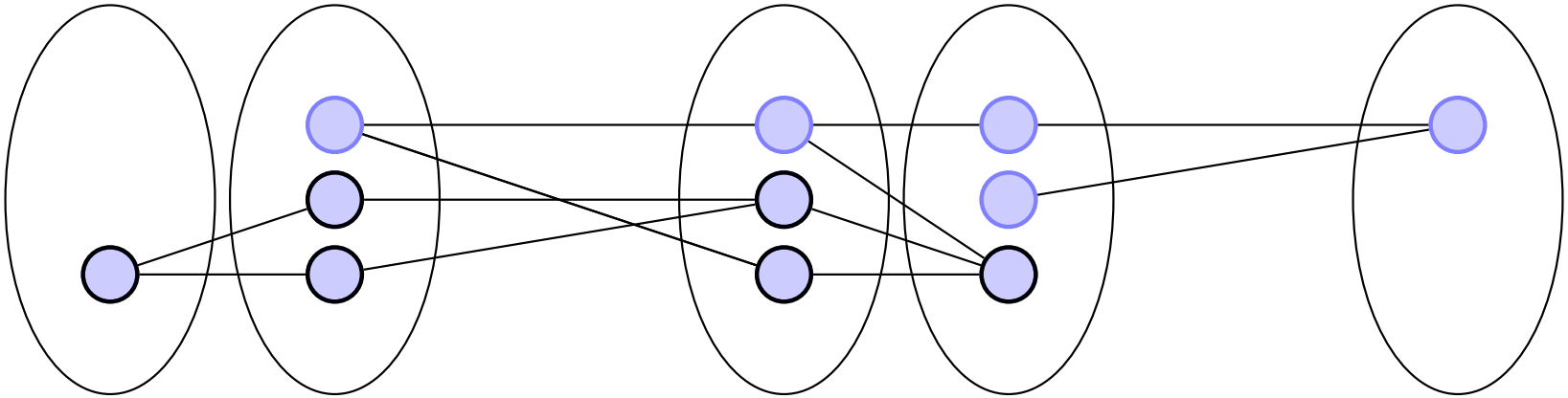

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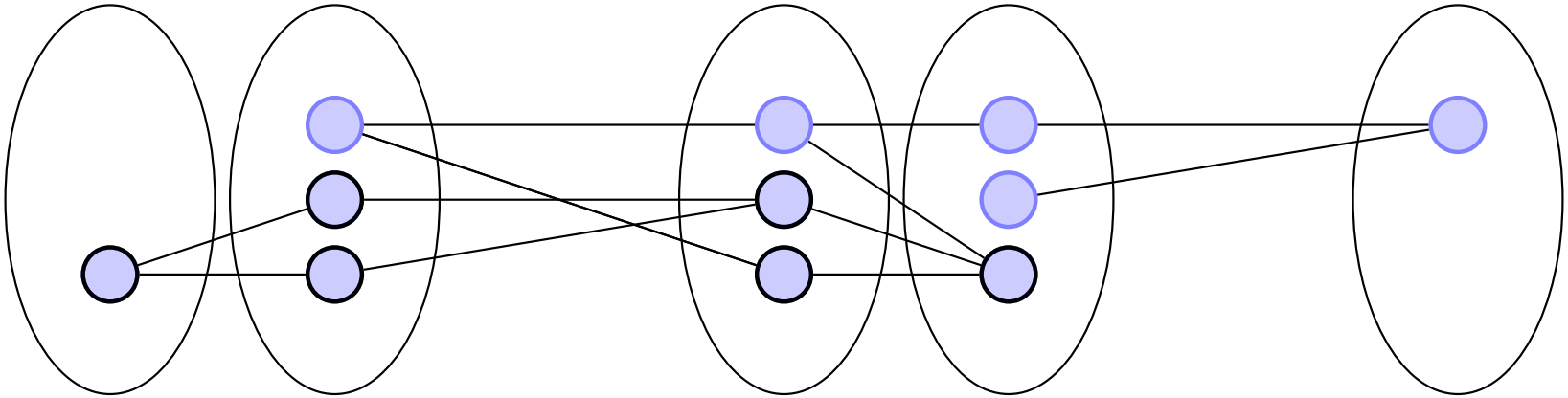

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- Additional trickery gives us a long part of the path instance where everything is subdirect.
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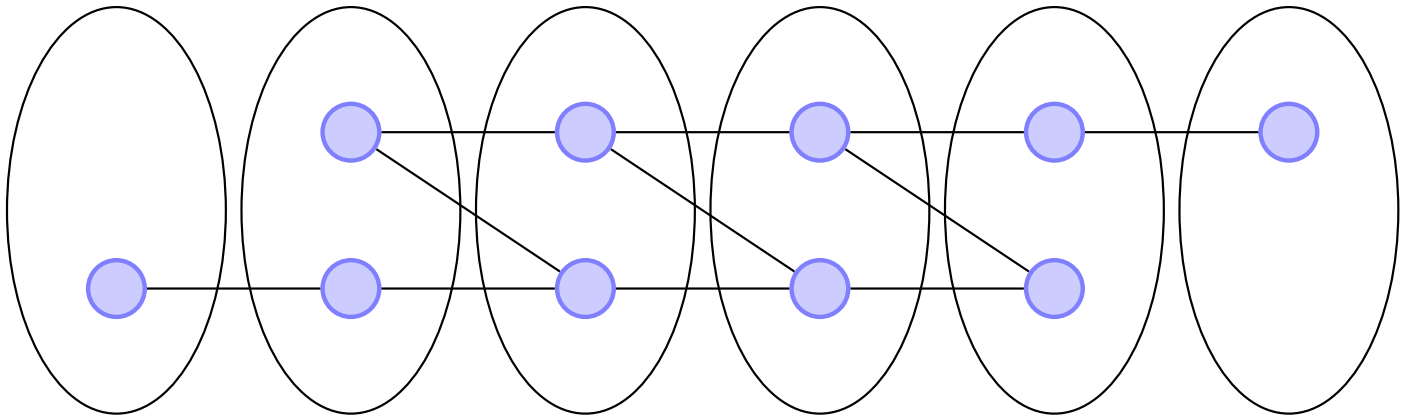

Summing up, part 1

Theorem

Given n-permutatable relational structure $\mathbb{A}$ such that CSP($\mathbb{A}$) *has bounded pathwidth duality, there is a symmetric Datalog program that decides* CSP($\mathbb{A}$).

- V. Dalmau and B. Larose previously proved this for 2-permutable structures.
- n-permutability is necessary by L. Egri, B. Larose, and P. Tesson.

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Algebraic view (due to L. Egri, B. Larose, and P. Tesson):

- If $\neg\,\mathrm{CSP}(\mathbb{A})$ is solvable by linear Datalog, then the algebra of $\mathbb{A}$ must be (join) semidistributive.
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Conjecture

Semidistributive = linear Datalog = NL.

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Semidistributive and n-permutable = symmetric Datalog = L.

- Our result plus first conjecture gives the second conjecture.

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Thank you for your attention.

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