Affine Consistency and the Complexity of Semilinear Constraints*

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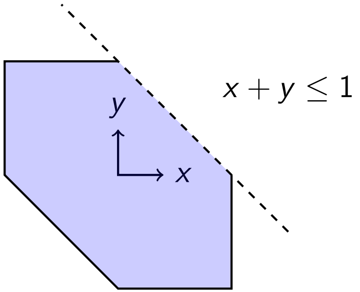
Outline

- Linear programming feasibility as a CSP
- Semilinear constraints: motivation and results
- Hard problems (NAE 3SAT)
- Easy problems (affine consistency)
- Future work

Linear Programming Feasibility and CSPs



LP feasibility: Given a set of (non-strict) inequalities over $\mathbb{Q}$, find a point that satisfies all of them.

As a constraint satisfaction problem (CSP):

- variables: x, y
- values: $\mathbb{Q}$
- constraints: $x + y \leq 1$, $x \leq 1$, ...

Linear Programming Feasibility and CSPs

- We can express LP feasibility as CSP(Γ), where Γ is the set of relations defined by linear inequalities of arbitrary arities.
- But we want a Γ with a finite signature.

Let $R_+ = \{(x, y, z) \in \mathbb{Q}^3 \mid x + y = z\}$.
Example. $x + y \leq 1$ can be pp-defined using $R_+, \leq, \{1\}$:

$$\exists z, w \in \mathbb{Q} : (x + y = z) \wedge (z \leq w) \wedge (w = 1)$$

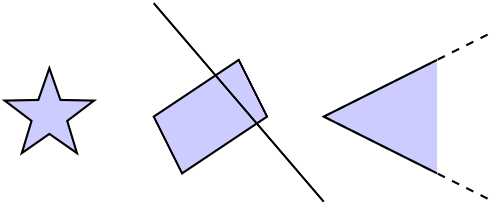
Proposition (Bodirsky, Jonsson, von Oertzen '12)

LP feasibility and CSP($\{R_+, \leq, \{1\}\}$) are polynomial-time equivalent problems.

Semilinear relations

- A relation is linear if it can be written as a finite intersection of strict and non-strict inequalities.
- A relation is semilinear if it can be written as a finite union of linear relations.

  (Alternatively, if it is first-order definable in $\{R_+, \leq, \{1\}\}$.)



Question

Let Γ be a finite set of semilinear relations. What is the computational complexity of CSP(Γ)?

Motivation

Theorem (Bodirsky, Jonsson, von Oertzen '12)

*Let Γ be a finite set of semilinear relations, $\{R_+, \leq, \{1\}\} \subseteq \Gamma$. CSP($\Gamma$) is either polynomial-time solvable or **NP**-complete.*

Tractability from essential convexity

Definition

A relation R is essentially convex if, for any points $p, q \in R$, it excludes at most a finite number of points from the line segment between p and q.

Theorem (Bodirsky, Jonsson, von Oertzen '12)

If Γ is a set of essentially convex semilinear relations, then $CSP(\Gamma)$ is in **P**.

- For any not essentially convex semilinear relation R, $\mathrm{CSP}(\{R_+, \leq, \{1\}, R\})$ is **NP**-hard.

A refined classification

Theorem (Bodirsky, Jonsson, von Oertzen '12)

Let Γ *be a finite set of semilinear relations,* $\{R_+, \leq, \{1\}\} \subseteq \Gamma$*.*
CSP(Γ) *is either polynomial-time solvable or* **NP***-complete.*

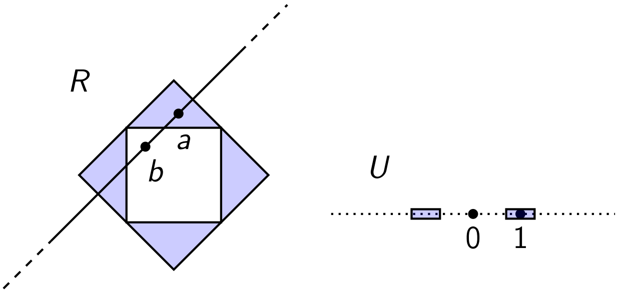
We relax the assumption on Γ:

Theorem (Jonsson, T. '14)

Let Γ *be a finite set of semilinear relations,* $\{R_+, \{1\}\} \subseteq \Gamma$*.*
CSP(Γ) *is either polynomial-time solvable or* **NP***-complete.*

We take a step towards removing $\{1\}$ from the assumption.

Gadgets



$$z \in U \equiv \exists x, y \in \mathbb{Q} : R(x, y) \wedge (x = a_x \cdot z + b_x \cdot (1 - z)) \\ \wedge (y = a_y \cdot z + b_y \cdot (1 - z))$$

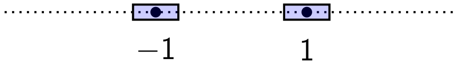
- Let $\langle \Gamma \rangle$ denote the set of all relations pp-definable in Γ.
  Example. $U \in \langle \{R, R_+, \{1\}\} \rangle$. Multiplications are simulated by R_+ and constants by R_+, $\{1\}$.

NP-completeness

The canonical **NP**-complete problem is Not-All-Equal 3SAT.
Equivalent with CSP($\{R_{NAE}\}$),

$$R_{NAE} = \{-1, 1\}^3 \setminus \{(-1,-1,-1),(1,1,1)\}$$

Let U be the following unary relation:



Consider the ternary relation $R \in \langle\{U, R_+\}\rangle$:

$$(x, y, z) \in R \equiv \exists w \in \mathbb{Q} : U(x) \wedge U(y) \wedge U(z) \wedge U(w) \wedge x + y + z + w = 0$$

The signs of a tuple in R encode a tuple in R_{NAE}.
$\implies$ CSP($\{R\}$) and CSP($\{R_{NAE}\}$) are equivalent.
$\implies$ CSP($\{U, R_+\}$) is **NP**-complete.

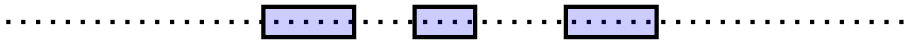
Bounded Non-constant Unary relations

Definition

A bnu is a unary relation U such that

- $U \subseteq (-M, M)$ for some $M < \infty$; and
- $|U| > 1$.

Example. A union of bounded intervals:



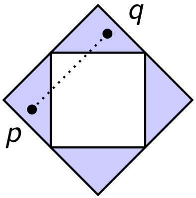
Example. Two points: $U = \{\frac{1}{2}, \frac{3}{2}\}$

Excluded intervals

Definition

A relation R excludes an interval if there exist points $p \neq q \in R$ and $0 < \delta_1 < \delta_2 < 1$ such that $p + (q - p) \cdot t \notin R$ for $\delta_1 \leq t \leq \delta_2$.

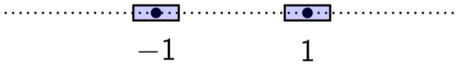
Example. The following relation excludes an interval:



Example. The relation $\mathbb{Q} \setminus \{0\}$ does not exclude an interval.

Sufficient conditions for NP-completeness

Recall the hard unary relation:



When does $\langle\Gamma\rangle$ contain such a relation?

The following conditions are together sufficient:

1. $\langle\Gamma\rangle$ contains a bnu U.
2. $\langle\Gamma\rangle$ contains a unary relation T that excludes an interval.

Proof idea. Use R_+ and $\{1\}$ to transform T so that the excluded interval is mapped to $(-\delta, \delta)$ for some $\delta \approx 1$, then transform U and use it to bound the resulting relation. □

Tractability

1. $\langle\Gamma\rangle$ contains a bnu U.
2. $\langle\Gamma\rangle$ contains a unary relation T that excludes an interval.

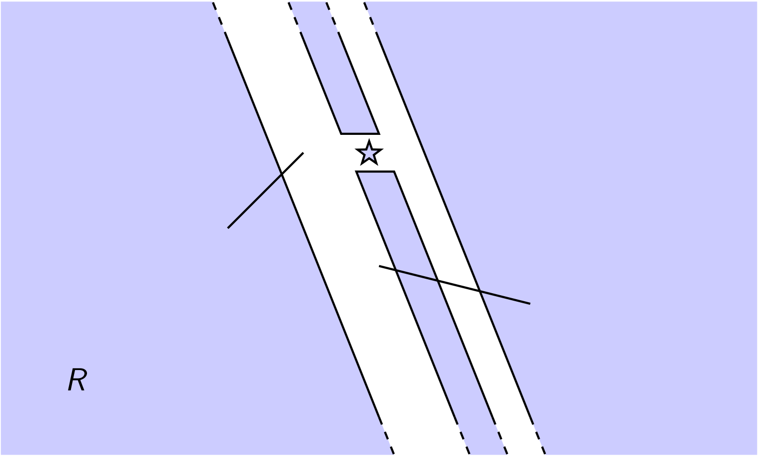
($\neg$2.) A semilinear relation is essentially convex iff it does not exclude an interval $\implies$ all relations in $\langle\Gamma\rangle$ are ess. convex.

($\neg$1.) Assume that $\langle\Gamma\rangle$ does not contain a bnu. Let $R \in \langle\Gamma\rangle$ be of arity k and L be a line in $\mathbb{Q}^k$. Then $R \cap L$ is

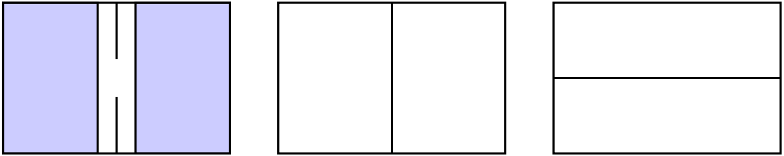
- empty; or
- a single point; or
- unbounded "in both directions" (L minus a bounded set).

No bnus

Example. $\langle R, R_+, \{1\}\rangle$ does not contain a bnu.



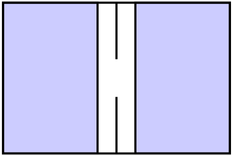
Algorithm



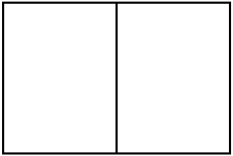
$R(x, y)$ $U(x)$ $U(y)$

A : upper bound on the solution space (initially, $A := \mathbb{Q}^2$)

Algorithm



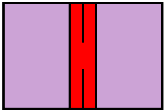
$R(x, y)$



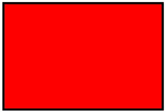
$U(x)$

$U(y)$

A : upper bound on the solution space (initially, $A := \mathbb{Q}^2$)

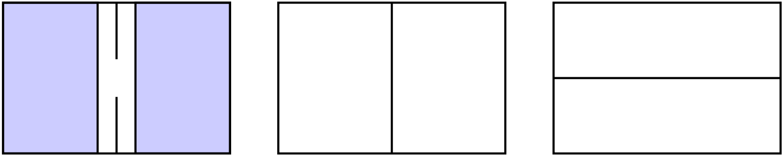


$R(x, y) \cap A$



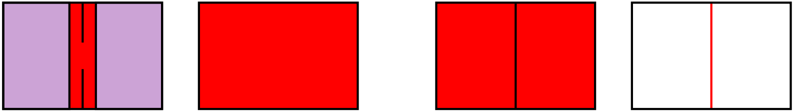
$A := \mathbb{Q}^2$

Algorithm



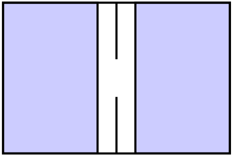
$R(x, y)$ $U(x)$ $U(y)$

A : upper bound on the solution space (initially, $A := \mathbb{Q}^2$)

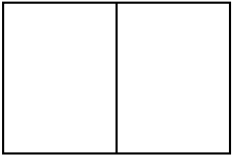


$R(x, y) \cap A$ $A := \mathbb{Q}^2$ $U(x) \cap A$ $A := \{x = 0\}$

Algorithm



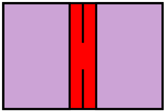
$R(x, y)$



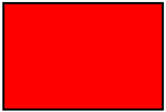
$U(x)$

$U(y)$

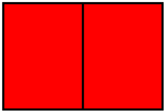
A : upper bound on the solution space (initially, $A := \mathbb{Q}^2$)



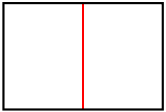
$R(x, y) \cap A$



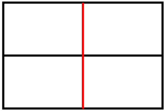
$A := \mathbb{Q}^2$



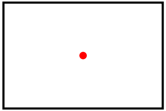
$U(x) \cap A$



$A := \{x = 0\}$

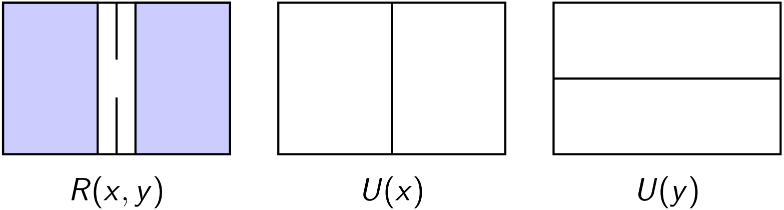


$U(y) \cap A$

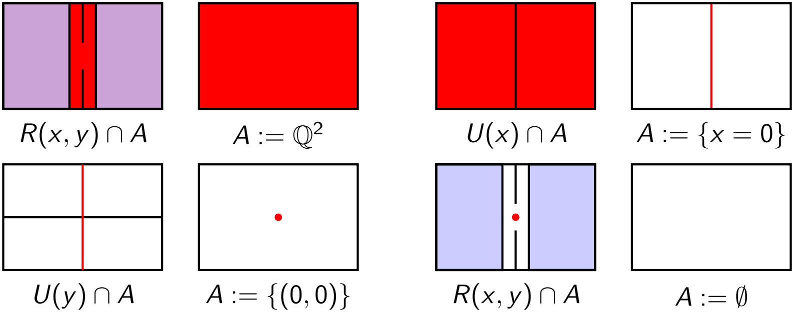


$A := \{(0, 0)\}$

Algorithm



A : upper bound on the solution space (initially, $A := \mathbb{Q}^2$)



Affine consistency

Consider an instance with variables $V = \{x_1, \ldots, x_n\}$ and constraints $\{R_i(x_{i1}, \ldots, x_{ik_i})\}_{i=1}^m$, $\{x_{i1}, \ldots, x_{ik_i}\} \subseteq V$

- Define $\hat{R}_i = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid (x_{i1}, \ldots, x_{ik_i}) \in R_i\}$
- affine subspace: translation of a linear subspace of $\mathbb{Q}^n$
- For $X \subseteq \mathbb{Q}^n$, let $\mathrm{aff}(X)$ denote the affine hull of X; the smallest affine subspace containing X

Definition

A set of constraints $\{R_i(x_{i1}, \ldots, x_{ik_i})\}_{i=1}^m$ is affinely consistent with respect to a (non-empty) affine subspace A if $\mathrm{aff}(\hat{R}_i \cap A) = A$ for all $1 \leq i \leq m$.

Tractability by affine consistency

Algorithm 1: Establish affine consistency

$A := \mathbb{Q}^n$
repeat
  foreach *constraint* $R_i(x_{i1}, \ldots, x_{ik_i})$ **do**
    $A := \mathrm{aff}(\hat{R}_i \cap A)$
  end
until *A does not change*
if $A \neq \emptyset$ **then** **return** "yes" **else** **return** "no"

Theorem

Let Γ be a finite set of semilinear relations, $\{R_+, \{1\}\} \subseteq \Gamma$. If there is no bnu in $\langle \Gamma \rangle$, then Algorithm 1 is correct and CSP(Γ) is in **P**.

Main points to prove:

- If "yes", then there exists a solution
- $\mathrm{aff}(\hat{R}_i \cap A)$ can be computed in polynomial time
- The representation size of A does not grow too fast

Result for $\{R_+, \{1\}\} \subseteq \Gamma$

Theorem (Jonsson, T. '14)

Let Γ be a finite set of semilinear relations, $\{R_+, \{1\}\} \subseteq \Gamma$.

- *If $\langle\Gamma\rangle$ does not contain a bnu, then CSP(Γ) is in* **P** *(affine consistency);*
- *otherwise, if no unary relation in $\langle\Gamma\rangle$ excludes an interval, then CSP(Γ) is in* **P** *(essential convexity);*
- *otherwise, CSP(Γ) is* **NP***-complete.*

First step towards removing $\{1\}$

(P_0) $\exists$ unary relation $U \in \langle\Gamma\rangle$ that contains a positive point and satisfies $U \cap (0, \epsilon) = \emptyset$ for some $\epsilon > 0$.

(P_∞) $\exists$ unary relation $U \in \langle\Gamma\rangle$ that contains a positive point and satisfies $U \cap (M, \infty) = \emptyset$ for some $M < \infty$.

Theorem (Jonsson, T. '14)

Let Γ be a finite set of semilinear relations, $\{R_+\} \subseteq \Gamma$ and assume that Γ satisfies (P_0) and (P_∞).

- *If $(0, \ldots, 0) \in R$ for all $R \in \Gamma$, then CSP(Γ) is in* **P** *(0-valid);*
- *otherwise, if $\langle\Gamma\rangle$ does not contain a bnu, then CSP(Γ) is in* **P** *(affine consistency);*
- *otherwise, if no unary relation in $\langle\Gamma\rangle$ excludes an interval, then CSP(Γ) is in* **P** *(reduction + essential convexity);*
- *otherwise, CSP(Γ) is* **NP***-complete.*

What's left?

- A relation R is homogeneous linear if it can be written as a finite intersection of strict and non-strict inequalities with 0 constant term.
- A relation R is homogeneous semilinear if it can be written as a finite union of homogeneous linear relations. (Alternatively, if it is first-order definable in $\{R_+, \leq\}$)

Theorem

Let Γ be a finite set of semilinear relations, $\{R_+\} \subseteq \Gamma$. If Γ fails to satisfy either (P_0) or (P_∞), then $CSP(\Gamma)$ is equivalent to $CSP(\Gamma')$ for a finite homogeneous semilinear language $\{R_+\} \subseteq \Gamma'$.

Future work

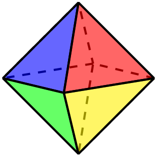
Problem

Classify the complexity of CSP(Γ) *for* Γ *that contain* R_+ *and are first-order definable in* $\{R_+, \leq\}$.

Without the restriction of containing R_+, this classification would have to contain:

- the 3-element CSP-classification (Bulatov '06), and
- the classification of all temporal constraint languages (Bodirsky and Kára '10).

Thank you for your attention!



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- Manuel Bodirsky, Peter Jonsson, and Timo von Oertzen
  Essential Convexity and Complexity of Semi-Algebraic Constraints
  Logical Methods in Computer Science 8(4) (2012)