Universal algebra and CSP Tutorial

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Outline

1. Basic objects
2. Avoiding structures
3. Maltsev conditions
4. Omitting types (Tame congruence theory)
5. Connection to CSP
6. Other buzzwords
   - Clones
   - Absorption

Basic Objects

Algebra: A structure $\mathbf{A} = (A, (f_i : i \in I))$ having functions, no relations.

Identity: Any sentence of the form $\forall \mathbf{x}[s(\mathbf{x}) = t(\mathbf{x})]$, s, t terms.

Equational class: Any class of algebras axiomatized by identities.

- Aka **variety**.
- Examples: {groups}, {abelian groups}, {R-modules}, etc.
- Varieties contain **free algebras**.

Special cases:

- **Idempotent** algebras/varieties: those for which every term s satisfies $s(x, x, \ldots, x) = x$ $(\forall x)$.
- **Finite** algebras (those with finite domain).
- $V(\mathbf{A}) = \mathrm{Mod}(\mathrm{Th}_{id}(\mathbf{A}))$, the variety **generated by A**.

Avoiding structures

Let $\mathbf{A} = (A, \ldots)$ be an algebra, and $\mathcal{V}$ a variety.

1. A relation $R \subseteq A^n$ is **compatible** with $\mathbf{A}$ if it is (the domain of) a subalgebra of $\mathbf{A}^n$.
2. A relational structure $\mathbb{A} = (A, (R_j : j \in J))$ is **compatible** with $\mathbf{A}$ if each of its relations R_j is compatible.
3. $\mathcal{V}$ **admits** a relational structure $\mathbb{B}$, if $\mathbb{B}$ is compatible with some $\mathbf{B} \in \mathcal{V}$; otherwise, $\mathcal{V}$ **avoids** $\mathbb{B}$.
4. $\mathcal{V}$ **avoids** a class $\mathcal{K}$ of relational structures if $\mathcal{V}$ avoids every $\mathbb{B} \in \mathcal{K}$.

Example

Example

The variety $\{\text{groups}\}$ avoids the class $\{\text{nontrivial posets}\}$.

Proof. Suppose $\mathbf{G}$ is a group and (G, E) is a poset compatible with $\mathbf{G}$.

This means E is a subgroup of $\mathbf{G}^2$.

Suppose $a, b \in G$ and aEb.

Then $(a, a), (a, b), (b, b) \in E$, so

$$(b, a) \;=\; (a, a)(a, b)^{-1}(b, b) \;\in\; E.$$

Thus aEb implies bEa.

Hence (G, E) is trivial. □

Theorem (Hagemann & Mitschke, 1973)

For any variety $\mathcal{V}$, tfae:

1. *$\mathcal{V}$ avoids $\mathcal{K} = \{$nontrivial posets$\}$.*
2. *For some $n \geq 1$, $\exists$ terms $p_1(x, y, z), \ldots, p_n(x, y, z)$ such that*

$$\mathcal{V} \models \forall xy \left[x = p_1(x, y, y) \;\&\; \bigwedge_{i=1}^{n-1} p_i(x, x, y) = p_{i+1}(x, y, y) \;\&\; p_n(x, x, y) = y \right].$$

Proof idea for (2) $\Rightarrow$ (1): Assuming $\mathbf{B} \in \mathcal{V}$, $\mathbf{E} \leq \mathbf{B}^2$, E is reflexive, and $(a, b) \in E$, evaluate the $p_i(x, y, z)$'s at $(a, a), (a, b), (b, b)$ to get

$$(b, c_1), (c_1, c_2), \ldots, (c_{n-1}, a) \in E.$$

The condition in (2) is an example of a **Maltsev condition**.

A **Maltsev condition** (on varieties) is a condition of the form

$$\bigvee_{n\in\omega} \underbrace{\left[\exists \underbrace{t_1, \ldots, t_{h(n)}}_{\text{terms}} \bigwedge_{i=1}^{k(n)} (\text{identities in the } t_j) \right]}_{C(n)}$$

Intuition: for "natural" classes $\mathcal{K}$, the varieties avoiding $\mathcal{K}$ "should" be characterized by a Maltsev condition, which "explains" why the varieties avoid $\mathcal{K}$.

Definition. A first-order sentence is a **finite product sentence** if it is true in $\mathbb{B}_1 \times \mathbb{B}_2$ whenever it is true in both factors.

Theorem (Taylor, 1973)

If $\mathcal{K}$ is definable by the negation of a finite product sentence, then "avoiding $\mathcal{K}$" is characterized by a Maltsev condition.

Many "classical" Maltsev conditions characterize the avoidance of "bad" configurations of compatible equivalence relations (aka **congruences**).

Example. $\mathcal{K}_{\neg DL}$ = the class of all structures $\mathbb{B} = (B, \alpha, \beta, \gamma)$ where

- α, β, γ are equivalence relations on B, and
- α, β, γ fail to satisfy the distributive law, i.e.,

$$\alpha \cap (\beta \vee \gamma) \neq (\alpha \cap \beta) \vee (\alpha \cap \gamma)$$

in the lattice of all equivalence relations on B.

Definition

A variety is **congruence distributive** (CD) if it avoids $\mathcal{K}_{\neg DL}$.

Theorem (Jónsson, 1967)

Being CD is characterized by a Maltsev condition.

Tame congruence theory

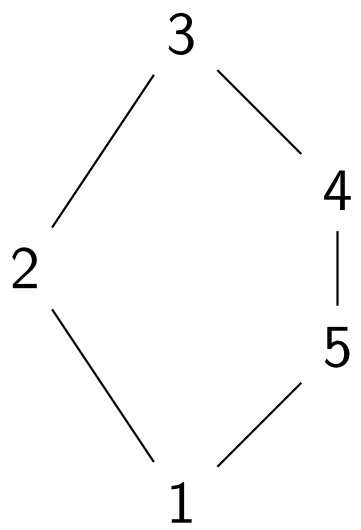
Developed by Hobby and McKenzie in the 1980s, about finite algebras.

- Discovers exactly 5 types of "local behaviour" of polynomial operations (i.e., terms with parameters).
- Called **type 1**, **type 2**,. . . ,**type 5**.
- If $\mathbf{A}$ is finite and $\mathcal{V} = V(\mathbf{A})$, can speak of $\mathcal{V}$ **omitting type i**.
- The types are naturally ordered:

For each **downset** S, it is natural to consider such $\mathcal{V}$ which omit S. E.g.,

- Omit $\{1\}$
- Omit $\{1, 2\}$
- Omit $\{1, 5\}$
- Omit $\{1, 2, 5\}$

etc.



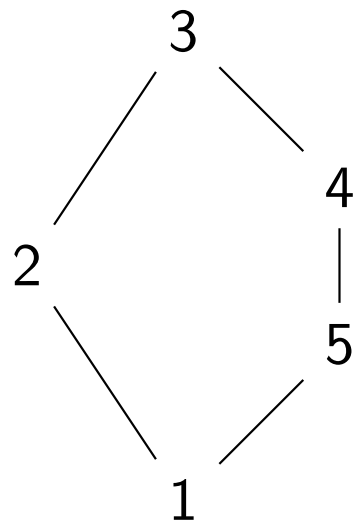
Theorem (Hobby & McKenzie, 1988)

For each downset S, among all varieties of the form $\mathcal{V} = V(\mathbf{A})$ with $\mathbf{A}$ finite, the varieties which omit S are characterized by a Maltsev condition.

- In each case, omitting S is characterized by avoiding a class of equivalence structures (i.e., by a condition on congruences).

  For example:

  - Omitting $\{1, 2\} \equiv$ "congruence meet semi-distributivity."
  - Omitting $\{1, 4, 5\} \equiv$ "congruence k-permutability for some k."



- If we restrict to **idempotent** $\mathcal{V}$, much simpler avoidance classes are known.

  (More later.)

Connection to CSP

Fix $\mathbb{A}$ – a finite relational structure in a finite signature.

The complexity of CSP($\mathbb{A}$) is governed by the $\mathbb{B}$ which pp-interpret in $\mathbb{A}$.

For example: let

$$\begin{aligned}
\mathbb{B}_{3SAT} &= (\{0,1\},\ \text{all 3-ary relations}) \\
\mathbb{B}_{Horn} &= (\{0,1\},\ \text{all 3-ary Horn relations}) \\
&= (\{0,1\},\ \text{all subalgebras of } (2,\wedge)^3) \\
\mathbb{B}_{\leq} &= (\{0,1\},\ \leq, \{0\}, \{1\}) \\
\mathbb{B}_{\mathbb{Z}_p,1} &= (\mathbb{Z}_p,\ \text{“}x+y=z\text{”},\ \{1\}).
\end{aligned}$$

1. If $\mathbb{B}_{3SAT}$ is pp-interpretable in $\mathbb{A}$, then CSP($\mathbb{A}$) is NP-complete.
2. If $\mathbb{B}_{Horn}$ is pp-interpretable in $\mathbb{A}$, then CSP($\mathbb{A}$) is P-hard.
3. If $\mathbb{B}_{\leq}$ is pp-interpretable in $\mathbb{A}$, then CSP($\mathbb{A}$) is NL-hard.
4. If $\mathbb{B}_{\mathbb{Z}_p,1}$ is pp-interpretable in $\mathbb{A}$, then CSP($\mathbb{A}$) is Mod$_p$-hard.

Characterizing pp-interpretability

Fix $\mathbb{A}$ – a finite relational structure in a finite signature.

Let $\mathbf{Pol}(\mathbb{A})$ be the "richest" algebra compatible with $\mathbb{A}$.

- $=$ "the polymorphism algebra of $\mathbb{A}$"

Theorem ($\approx$ Bodnarčuk *et al*, 1969)

For any finite structure $\mathbb{B}$,

$$\mathbb{B} \text{ is pp-interpretable in } \mathbb{A} \iff \mathbb{B} \text{ is compatible with some } \mathbf{B} \in V(\mathbf{Pol}(\mathbb{A})).$$

$$\iff V(\mathbf{Pol}(\mathbb{A})) \text{ admits } \mathbb{B}.$$

Hence the complexity of CSP($\mathbb{A}$) is determined by the structures which $V(\mathbf{Pol}(\mathbb{A}))$ admits/avoids.

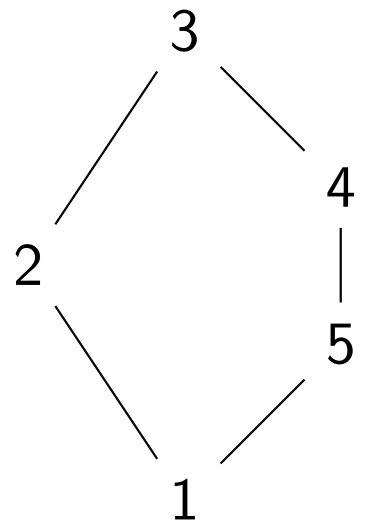
The idempotent case

Standard CSP assumption: $\mathbb{A}$ "has all constants" (as unary relations).

Hence $V(\mathbf{Pol}(\mathbb{A}))$ is **idempotent**: every f satisfies $f(x, x, \ldots, x) = x$.

Returning to Tame Congruence Theory: in the idempotent case,

1. $\mathcal{V}$ omits $\{1\}$ iff $\mathcal{V}$ avoids $\mathbb{B}_{3SAT}$.
2. $\mathcal{V}$ omits $\{1, 5\}$ iff $\mathcal{V}$ avoids $\mathbb{B}_{Horn}$.
3. $\mathcal{V}$ omits $\{1, 4, 5\}$ iff $\mathcal{V}$ avoids $\mathbb{B}_{\leq}$, iff $\mathcal{V}$ avoids {nontrivial posets}.
4. $\mathcal{V}$ omits $\{1, 2\}$ iff $\mathcal{V}$ avoids $\mathcal{K}_{Ab}$ where



$\mathcal{K}_{Ab} = \{\mathbb{B}_{G,b} : (G, +)$ an abelian group, $b \neq 0\}$.

The bad structures coincide with the (known) hard structures for CSP!

Algebraic Dichotomy Conjecture

Assume that $\mathbb{A}$ "has constants." Let $\mathcal{V} = V(\mathbf{Pol}(\mathbb{A}))$. $\mathcal{V}$ is idempotent.

$$\begin{array}{lcl} \text{CSP}(\mathbb{A}) \text{ is hard (NP-complete)} & \text{if} & \mathbb{B}_{3SAT} \text{ is pp-interpretable in } \mathbb{A} \\ & \iff & \mathcal{V} \text{ admits } \mathbb{B}_{3SAT} \\ & \iff & \mathcal{V} \text{ does not omit } \{1\}. \end{array}$$

We can't think of any other reason for $\text{CSP}(\mathbb{A})$ to be hard.

Algebraic Dichotomy Conjecture (Bulatov, Krokhin & Jeavons, 2005)

For finite $\mathbb{A}$ with constants, if $\mathcal{V}$ omits $\{1\}$, then $\text{CSP}(\mathbb{A})$ is in P.

Related conjectures:

1. If $\mathcal{V}$ omits $\{1, 2, 5\}$, then $\text{CSP}(\mathbb{A})$ is in NL.
2. If $\mathcal{V}$ omits $\{1, 2, 4, 5\}$, then $\text{CSP}(\mathbb{A})$ is in L.

Maltsev conditions for "omitting $\{1\}$"

Suppose $\mathcal{V} = V(\mathbf{A})$ with $\mathbf{A}$ finite. $\mathcal{V}$ omits $\{1\}$ iff $\mathcal{V}$ has ...

1. (Taylor) For some $n \geq 2$, a **Taylor** term $T(x_1, \ldots, x_n)$: idempotent and satisfies identities incompatible with T being a projection.
2. (Hobby, McKenzie) For some $n \geq 0$, terms $f_1(x, y, z), \ldots, f_n(x, y, z)$, $d(x, y, z)$ satisfying identities implying that $d(x, y, z)$ looks like $x - y + z$ on any block of any "abelian" congruence (in any member of the variety).
3. (Maróti, McKenzie) For some $n \geq 2$, a **weak near unanimity** (WNU) term $w(x_1, \ldots, x_n)$: idempotent and

$$w(y, x, x, \ldots, x) = w(x, y, x, \ldots, x) = \cdots = w(x, \ldots, x, y).$$

4. (Barto, Kozik) For some $n \geq 2$, a **cyclic** term operation of arity n.
5. (Siggers, with improvements) A 4-ary idempotent term $s(x, y, z, w)$ satisfying $\forall xyz[s(x, y, z, x) = s(y, x, y, z)]$.

Clones

Fix an algebra **A** (finite or infinite).

For $n \geq 1$ let $\mathrm{Clo}_n(\mathbf{A}) = \{\text{all term functions of } \mathbf{A} \text{ of arity } n\} \subseteq A^{A^n}$.

Definition

The **clone algebra** of **A** is the many-sorted algebra **Clo**(**A**), having

- Sorts $S_1, S_2, \ldots, S_n, \ldots$ where $S_n = \mathrm{Clo}_n(\mathbf{A})$.
- For $m, n \geq 1$, a "composition operation"

$$C_m^n : \; S_n \times \underbrace{S_m \times \cdots \times S_m}_{n} \to S_m.$$

- For $n \geq 1$, constants $e_1^n, \ldots, e_n^n \in S_n$ naming the n projection maps.

Fact: $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Clo}(\mathbf{B})$ iff $V(\mathbf{B})$ satisfies every Maltsev condition satisfied by $V(\mathbf{A})$.

Prague Absorption

Prehistory: Bulatov (2006), Kiss & Valeriote (2007).
Godfathers: Barto & Kozik (2009, 2010, 2012, 2014).

The idea, roughly:

- Take your favourite idempotent linear Maltsev condition $\mathcal{C} = \bigvee_n C(n)$, whose identities include some of the form $f(x, \ldots, x, y, x, \ldots, x) = x$.
- Given a finite idempotent algebra $\mathbf{A}$ and subalgebra $\mathbf{B} \leq \mathbf{A}$: say $\mathbf{B}$ $\underline{\mathcal{C}\text{-absorbs}}$ $\mathbf{A}$ if for some n, $\mathbf{A}$ has terms which "satisfy $C(n)$ modulo B" in the following sense:
  - Identities not of the above form are satisfied in the usual sense.
  - For each identity of the form $f(x, \ldots, x, y, x, \ldots, x) = x$, require only that $f(B, \ldots, B, A, B, \ldots, B) \subseteq B$.
- Existence of proper absorbing algebras typically allows badness to be "shrunk." Absence of proper absorbing algebras imply some of the consequences of satisfying $\mathcal{C}$.

See e.g. Barto, Kozik & Stanowsky (2015) for some recent magic.

In closing

I was offered two tuturial hours.

You're welcome!