

Constrained Hawkes processes for modeling limit order books.

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Outline

- 1 Limit order books
- 2 Constrained Hawkes processes
- 3 Some applications

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High frequency price data

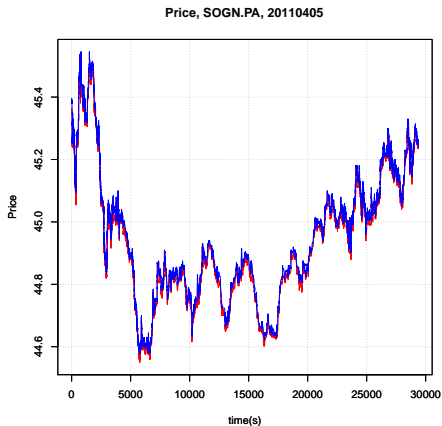
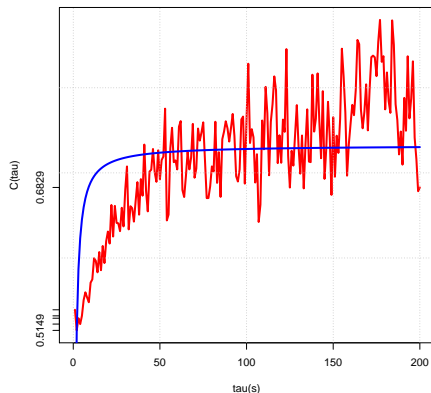


Figure: Price of an asset over **one day**.

Signature plots

How do the usual models behave at small scales ?

Signature Plot, BestAsk, SOGN.PA, 20110405



Signature Plot, BestBid, SOGN.PA, 20110405

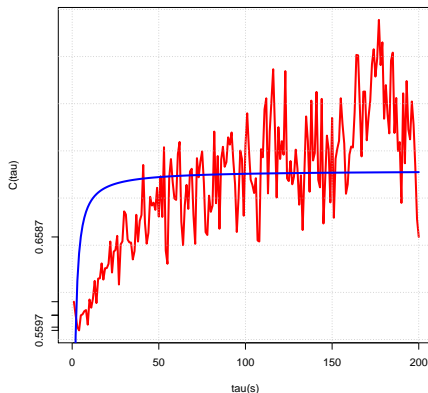


Figure: Red: Normalized realized volatility as a function of the sampling period.

High frequency price data: another asset

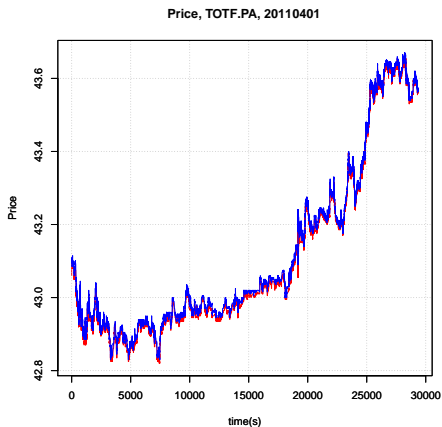


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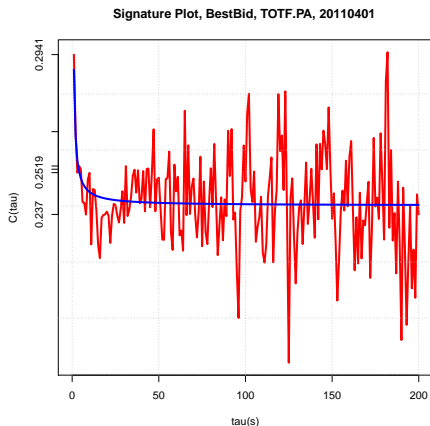
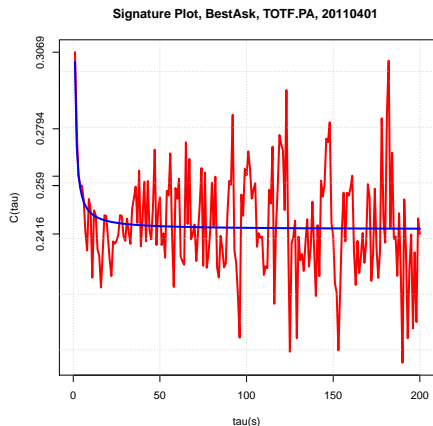


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The smallest time scale: Limit order book

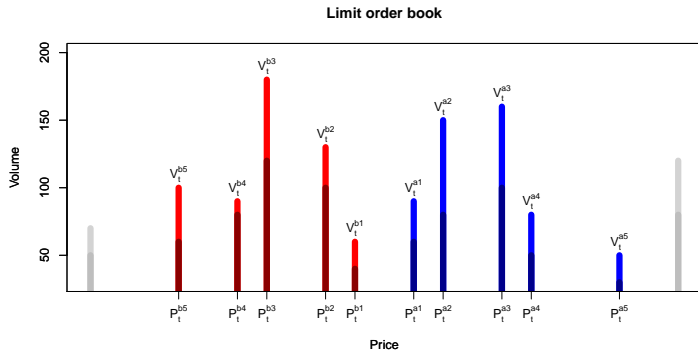


Figure: A limit order book (LOB) at a given fixed time. Bid prices (red) and Ask prices (blue) available for market orders.

Limit order book events: limit order arrival

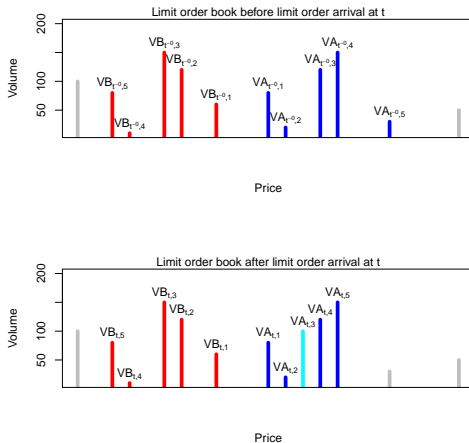


Figure: LOB before and after a limit order. Light blue: new ask limit.

Limit order book events: limit order cancellation

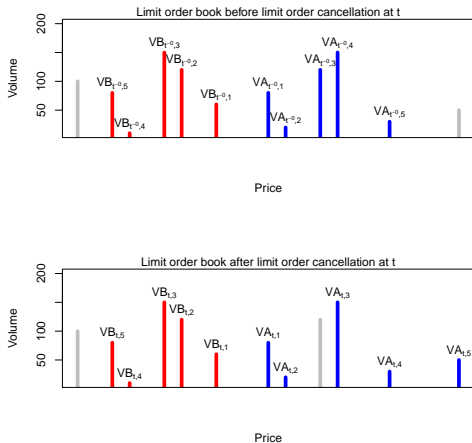


Figure: LOB before and after a limit order cancellation. Gray: canceled ask limit.

Time evolution of a LOB and mid-price.

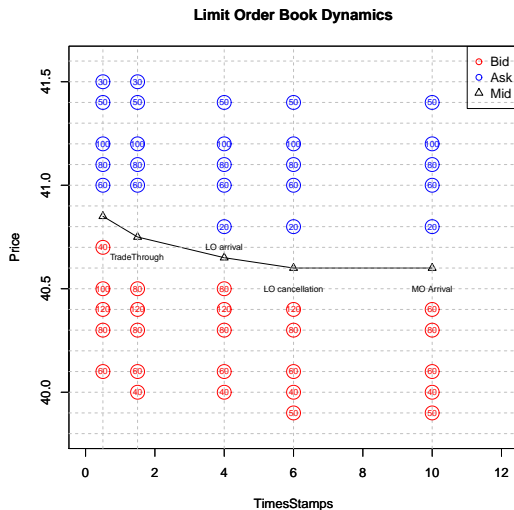
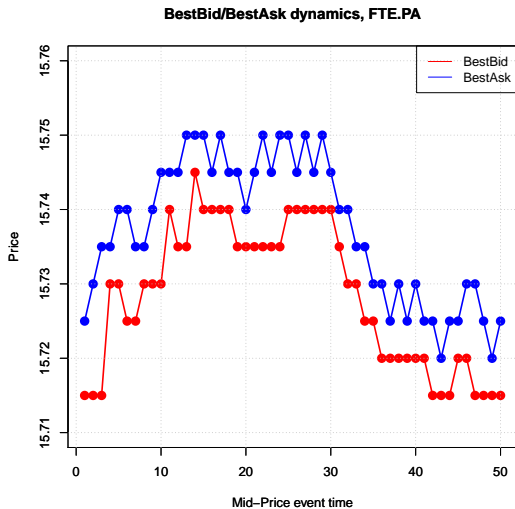


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A simplified LOB: Best Bid (BB) and Best Ask (BA) prices.



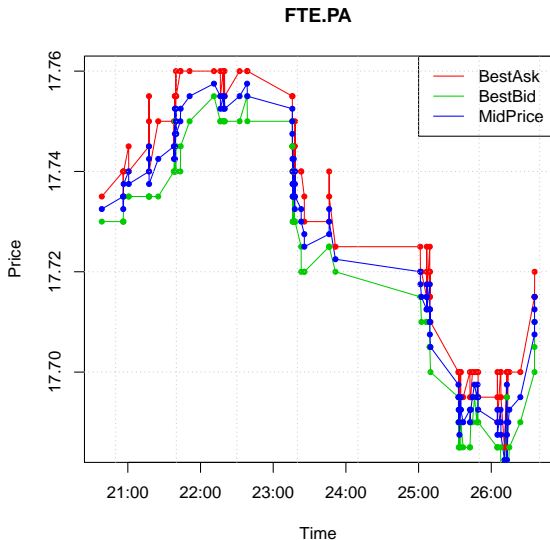


Figure: Successive BB and BA events in physical time (mn).

Point process of a simplified LOB

We consider the marked point process describing the dynamics of the BB and BA prices,

$$N = \sum_k \delta_{T_k, I_k} \quad \text{with} \quad 0 < T_1 < T_2 < \dots \quad \text{and} \quad I_1, I_2, \dots \in \{1, \dots, p\},$$

where each mark i in $\{1, \dots, p\}$ corresponds to a quantified shift of either the BB or the BA price, e.g.

- ▷ $i = 1$ Best Ask price moves upward one tick,
- ▷ $i = 2$ Best Ask price moves downward one tick,
- ▷ $i = 3$ Best Bid price moves upward one tick,
- ▷ $i = 4$ Best Bid price moves downward one tick.

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Extensions

By increasing the set of marks, one can consider one marked process describing the LOBs of [several assets](#).

Point process and prices dynamics

One can recover the dynamics of **BB**, **BA** and **mid** prices from the point process N through formulas of the form

$$P_t - P_0 = N \left(\mathbb{1}_{(0,t]} \otimes J \right) = \sum_{0 < T_k \leq t} J(I_k), \quad t > 0.$$

For instance, in the previous example,

$$J(1) = 1, J(2) = -1, J(3) = J(4) = 0$$

corresponds to $P_t = \text{BA price}$.

Point process and BB-BA spread

The gap between the BB price and the BA price is called the **spread**, from now on denoted by

$$S_t = \text{BA price}_t - \text{BB price}_t \in \{1, 2, 3, \dots\}.$$

We will denote by J the corresponding function on $\{1, \dots, p\}$ such that, for all $t > 0$,

$$S_t = S_0 + N(\mathbb{1}_{(0,t]} \times J) = S_0 + \sum_{0 < T_k \leq t} J(I_k).$$

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Important remarks

- ▶ J takes positive and negative values while S only takes positive ones.
- ▶ S_t behaves as a stationary random process.
- ▶ BB and BA prices typically behave as integrated (and thus co-integrated) stationary processes.

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Hawkes processes

Consider a marked point process $N = \sum_k \delta_{T_k, I_k}$ with

$\dots < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \dots$ and $\dots, I_{-1}, I_0, I_1, I_2, \dots \in \mathcal{I}$.

It is an Hawkes process if its conditional density is of the form

$$\mu(t, i) = \mu_0(i) + \int_{(-\infty, t)} \phi(t - s, j; i) N(ds, dj),$$

where $\mu_0 : \mathcal{I} \rightarrow \mathbb{R}_+$ is called the **immigrant intensity** and $\phi : [0, \infty) \times \mathcal{I}^2 \rightarrow \mathbb{R}_+$ is called the **fertility** function.

Multivariate Hawkes processes

If $\mathcal{I} = \{1, \dots, p\}$, The marked Hawkes process can be seen as a **multivariate Hawkes** process

$$N_i = N(\cdot \times \{i\}), \quad 1 \leq i \leq p,$$

the fertility is written as a $p \times p$ matrix $\phi(t) = [\phi_{i,j}(t)]_{i,j}$,

$$\mu(t, i) = \mu_0(i) + \int_{(-\infty, t)} \sum_{j=1}^p \phi_{i,j}(t-s) N_j(ds), \quad 1 \leq i \leq p,$$

or in a more compact form

$$\mu(t) = \mu_0 + \int_{(-\infty, t)} \phi(t-s) \mathbf{N}(ds).$$

Basic stability condition

It can be shown that such a point process is well defined and admit a stationary version if

(BC) the spectral radius of the $p \times p$ matrix $\mathfrak{N} = [\alpha_{i,j}]_{i,j} = \int_0^\infty \phi(t) dt$ is in $[0, 1)$.

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- ▶ A Hawkes process can be represented as a **cluster process**.
- ▶ It can be generalized to **spatial** point processes.
- ▶ It can be easily **simulated** using such representation or Ogata's algorithm (for temporal Hawkes processes).
- ▶ **Second order properties** (intensity, covariance measure, Bartlett spectrum) are well known.
- ▶ The delay and mark of the next event can be forecasted using the **Hazard rate**.
- ▶ Hawkes processes can be made **locally stationary** (ongoing work).

Hawkes processes for modeling a simple LOB

Suppose that a stationary Hawkes process N is used to model the dynamics of a simple LOB as defined previously yielding to

$$P_t - P_0 = N \left(\mathbb{1}_{(0,t]} \otimes J \right) = \sum_{0 < T_k \leq t} J(I_k), \quad t > 0 .$$

for a BB, BA or mid-price P (with an adequate J) and

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$$S_t = S_0 + N \left(\mathbb{1}_{(0,t]} \times J \right) = S_0 + \sum_{0 < T_k \leq t} J(I_k).$$

However, for a Hawkes process, provided that $\mu_0(i) > 0$ for all i , we have, for any k ,

$$\min_{1 \leq i \leq p} \mathbb{P}(I_{k+1} = i \mid \mathcal{F}_{T_k}) > 0,$$

where $\mathcal{F}_t = \sigma(N(A \times \{i\}), A \in \mathcal{B}((-\infty, t]), i \in \{1, \dots, p\})$.

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In other words, the BB and BA prices do behave as integrated processes but not as **cointegrated** ones.

Idea :

modify the conditional density by adding **constraints** depending on S_t .

Constrained Hawkes processes

We consider a point process N with marks in $\{1, \dots, p\}$ with conditional intensity given by, for all $i = 1, \dots, p$,

$$\mu(t, i) = \begin{cases} 0 & \text{if } \mathbf{S}(t-) \in \mathbf{A}_i \\ \mu_0(i) + \int_{(-\infty, t)} \sum_{j=1}^p \phi_{i,j}(t-s) N(ds \times \{j\}) & \text{otherwise,} \end{cases}$$

where \mathbf{S} is a q -dimensional process valued in \mathbb{N}^q and defined by

$$\mathbf{S}_t = \mathbf{S}_0 + N(\mathbb{1}_{(0,t]} \times \mathbf{J}),$$

for some $\mathbf{J} : \{1, \dots, p\} \rightarrow \mathbb{Z}^q$.

Here

- ▷ p denotes the number of marks
- ▷ q denotes the number of constraints.
- ▷ $\mathbf{A}_1, \dots, \mathbf{A}_p$ are constraints subsets of \mathbb{Z}^q .

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- ▶ Application to LOB modeling.

A very special case

Consider the simple LOB process, so that

- ▷ $p = 4$, $q = 1$,
- ▷ S_t is the spread at time t and, at each event, moves a tick upward or downward,
- ▷ $A_i = \{1\}$ for the events i making the spread move downward, so that S_t remains positive.

Take moreover the simple case $\phi = 0$ (no memory case : the conditional density does not depend on N).

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Take moreover the simple case $\phi = 0$ (no memory case : the conditional density does not depend on N).

Then S_t alone is a birth and death process on \mathbb{N} and the ergodicity is equivalent to

$$\mathbf{J}^T \boldsymbol{\mu}_0 < 0 .$$

(negative drift)

Markov assumption

Let us investigate the case where

$$\phi_{i,i}(t) = \alpha_{i,j} \beta e^{-\beta t}, \quad t \geq 0,$$

so that the unknown parameters are reduced to $\mathfrak{N} = [\alpha_{i,j}]$, $\beta > 0$ and $\mu_0 \in (0, \infty)^p$. Then, defining the \mathbb{R}^p valued process

$$\lambda(t) = \int_{-\infty}^t \phi(t-s) \mathbf{N}(ds),$$

we have that $\mathbf{X}(t) = (\mathbf{S}(t), \lambda(t))$ is a Markov process (due to the exponential form of the fertility function).

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we have that $\mathbf{X}(t) = (\mathbf{S}(t), \lambda(t))$ is a Markov process (due to the exponential form of the fertility function).

Moreover, the following discrete time processes are Markov chains :

- ▷ $\mathbf{X}_k = \mathbf{X}(T_k)$, with Markov kernel Q on $X = \mathbb{N}^q \times (0, \infty)^p$,
- ▷ $\mathbf{Y}_k = (I_k, \mathbf{X}_k)$, with Markov kernel \check{Q} on $Y = \{0, \dots, p\} \times X$,
- ▷ $\mathbf{Z}_k = (\Delta_k, I_k, \mathbf{X}_k)$, where $\Delta_k = T_k - T_{k-1}$, with Markov kernel \bar{Q} on $Z = \mathbb{R}_+ \times Y$.

Irreducibility, aperiodicity, partial drift

Some conditions on \mathfrak{N} and \mathbf{J} are required to get that

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- ▶ the above chains are ψ -irreducible and aperiodic (by adding an artificial mark $i = 0$ such that $\mathbf{J}(0) = 0$).
- ▶ For all $K = 1, 2, 3, \dots$ and $M > 0$, all sets $\{1, \dots, K\}^q \times (0, M]^p$ are petite-sets for Q .
- ▶ we have the partial drift condition,

$$[Q(\mathbb{1}_{\mathbb{Z}_+^q} \otimes V_{1,\gamma})](\mathbf{s}, \boldsymbol{\ell}) \leq \theta V_{1,\gamma}(\boldsymbol{\ell}) + b \mathbb{1}_{(0,M]^p}(\boldsymbol{\ell}),$$

where $M, b > 0$, $\theta \in (0, 1)$ and

$$V_{1,\gamma}(\boldsymbol{\ell}) = e^{\gamma \mathbf{u}^T \boldsymbol{\ell}},$$

for some $\gamma > 0$ and \mathbf{u} some vector with positive entries.

Geometric ergodicity: case $q = 1$

Consider the case $q = 1$ and suppose that

$$\mathbf{J}^T (I - \mathbb{N})^{-1} \boldsymbol{\mu}_0 < 0 .$$

This actually means that, would the constraints be removed, the process J would have a **negative drift under the stationary distribution** (and thus be eventually negative with probability 1).

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Consider the case $q = 1$ and suppose that

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This actually means that, would the constraints be removed, the process J would have a **negative drift under the stationary distribution** (and thus be eventually negative with probability 1).

Then we obtain a **complete drift condition** which implies that Q is $(V_{0,\gamma_0} \otimes V_{1,\gamma_1})$ -**geometrically ergodic** for some $\gamma_0, \gamma_1 > 0$, with

$$V_{0,\gamma_0}(s) = e^{\gamma_0 s}$$

and V_{1,γ_1} defined as above.

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- ▶ Scaling limit (Donsker Theorem)

$$T^{-1/2} (P_{tT} - P_0 - tT\mathbb{E}^0[J])_{t \in [0,1]} \Rightarrow \sigma(J) (B_t)_{t \in [0,1]} \quad \text{in } D, \quad (1)$$

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where B is the standard Brownian motion.

- ▶ we expect $\mathbb{E}^0[J] = 0$ for the BB or BA price, and P_t behaves as a random walk [at large scales](#).

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$$T^{-1/2} (P_{tT} - P_0 - tT\mathbb{E}^0[J])_{t \in [0,1]} \Rightarrow \sigma(J) (B_t)_{t \in [0,1]} \quad \text{in } D, \quad (1)$$

where B is the standard Brownian motion.

- ▶ we expect $\mathbb{E}^0[J] = 0$ for the BB or BA price, and P_t behaves as a random walk [at large scales](#).
- ▶ It is of course not the case for S which means that $\sigma(J) = 0$.

Consequences

All the usual good properties of geometrically ergodic Markov chains follow :

- ▶ Ergodicity, central limit theorems for the chains Q , \check{Q} , \bar{Q} ,
- ▶ Ergodicity, central limit theorems for any continuous time processes

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- ▶ It is of course not the case for S which means that $\sigma(J) = 0$.
- ▶ The result [can be extended to all \$q \geq 1\$](#) by [recursively](#) checking negative drifts on the chains obtained by removing an arbitrary set of constraints.

- 1 Limit order books
- 2 Constrained Hawkes processes
- 3 Some applications**

Simple LOB

We use the **Constrained Hawkes process** to describe the dynamics of a simple LOB using the marks

- ▷ $i = 1$ Best Ask price moves upward one tick,
- ▷ $i = 2$ Best Ask price moves downward one tick,
- ▷ $i = 3$ Best Bid price moves upward one tick,
- ▷ $i = 4$ Best Bid price moves downward one tick.

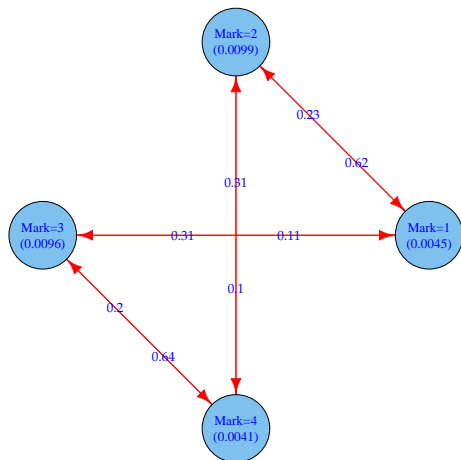
In this case we have

- ▷ $p = 4$, $q = 1$, S_t is the spread at time t .

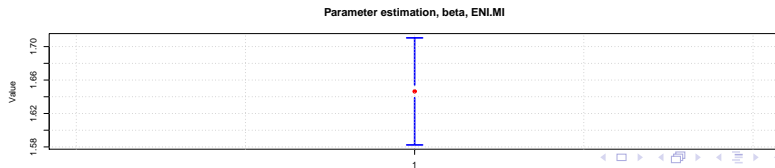
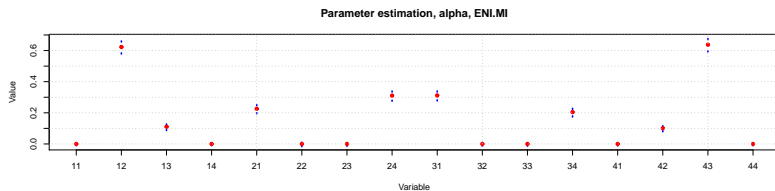
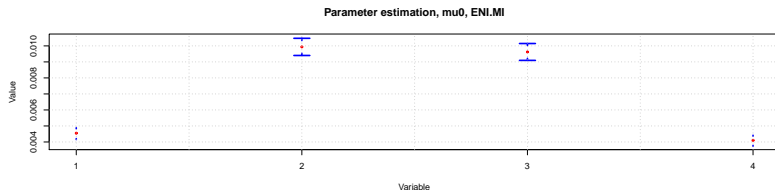
All the parameters are estimated by numerically **maximizing the likelihood**.

Excitation and immigrant intensities for ENI.MI, over ten days, time unit = seconds

ScLOBHP, Cross-Excitation Map, ENI.MI

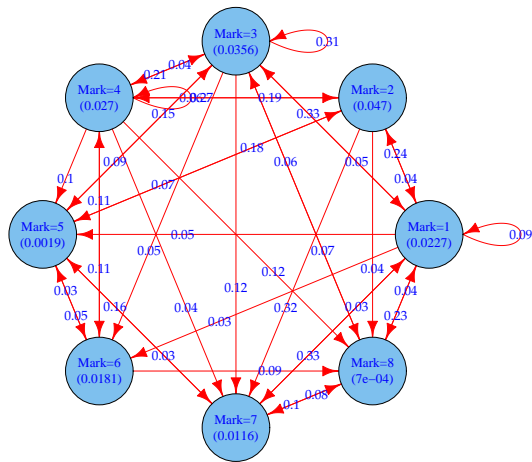


All parameters, same data, same unit

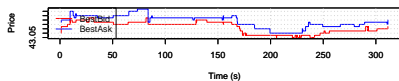


One/two ticks events, TOTF.PA

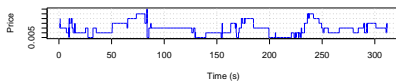
ScLOBHP, Cross-Excitation Map, TOTF.PA



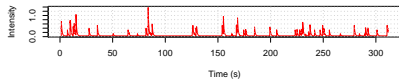
Price dynamics in limit Order Book



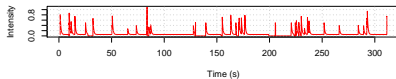
The dynamics of Spread



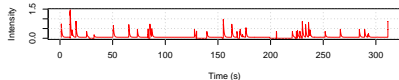
ScLOBHP Intensity, type = 1



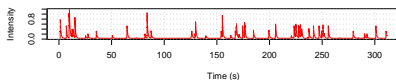
ScLOBHP Intensity, type = 2



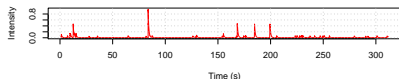
ScLOBHP Intensity, type = 3



ScLOBHP Intensity, type = 4



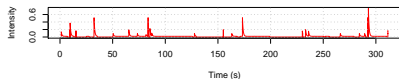
ScLOBHP Intensity, type = 5



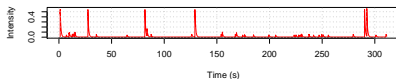
ScLOBHP Intensity, type = 6



ScLOBHP Intensity, type = 7



ScLOBHP Intensity, type = 8



Cross excitation, LOB with two assets

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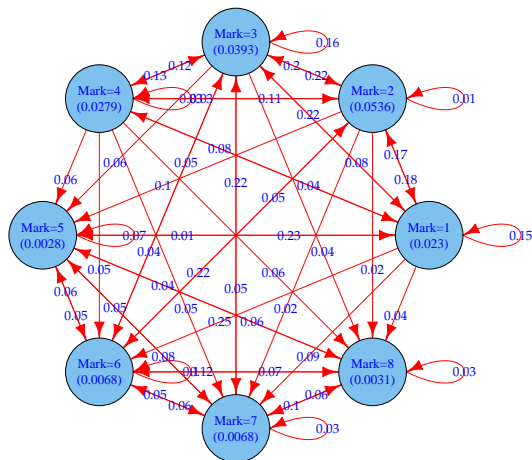
- ▶ $i = 1, 5$ Best Ask price moves upward one tick for asset 1,2, respectively,
- ▶ $i = 2, 6$ Best Ask price moves downward one tick for asset 1,2, respectively,
- ▶ $i = 3, 7$ Best Bid price moves upward one tick for asset 1,2, respectively,
- ▶ $i = 4, 8$ Best Bid price moves downward one tick for asset 1,2, respectively.

In this case we have

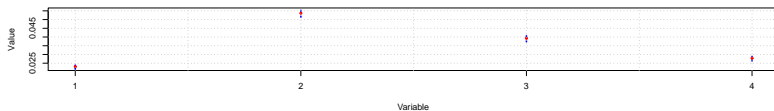
- ▶ $p = 8$, $q = 2$, \mathbf{S}_t contains the spreads of the two assets at time t .

Excitation and immigrant intensities for ENI.MI and TOTF.PA

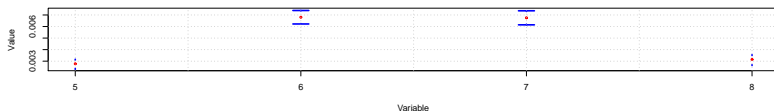
ScLOBHP, Cross-Excitation Map, TOTF.PA-ENI.MI



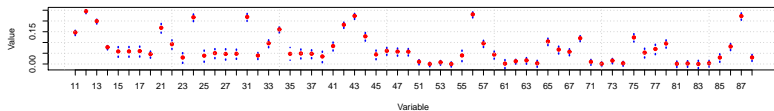
Parameter estimation, μ_0 , TOTF.PA



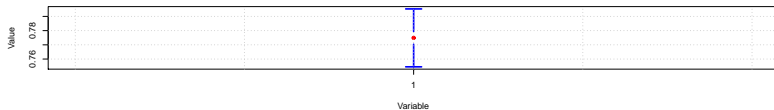
Parameter estimation, μ_0 , ENI.MI



Parameter estimation, α , TOTF.PA-ENI.MI



Parameter estimation, β , TOTF.PA-ENI.MI



Back to the case $q = 1$, conclusion

Using the estimated parameters one can evaluate the drift appearing in the stability condition :

$$\mathbf{J}^T (I - \mathbb{N})^{-1} \boldsymbol{\mu}_0 .$$

It seems to be a good indicator of the **volatility**.

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- ▶ Markov assumption should not be necessary.
- ▶ **Locally stationary** case (work in progress for standard Hawkes processes).

Further reading

Ban Zheng, François Roueff, and Frédéric Abergel. Modelling bid and ask prices using constrained Hawkes processes: Ergodicity and scaling limit. *SIAM J. Finan. Math.*, 5(1):99–136, February 2014. doi: 10.1137/130912980. Preprint available at [HAL] or [arXiv].