Multivariate limit theorems involving short-range and long-range dependence

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Outline

- Generalized Hermite processes
- Multivariate limit theorems involving generalized Hermite processes

Generalized Hermite processes

Hermite processes

The Hermite processes are limits of long-memory processes.

They can be represented as

$$Z(t) = \int_{\mathbb{R}^k}' \left\{ \int_0^t \prod_{j=1}^{k} (s - x_j)_+^{d-1} ds \right\} W(dx_1) \ldots W(dx_k),$$

The Hermite process $Z(t)$:

- is well-defined for $d \in \left(\frac{1}{2}(1 - \frac{1}{k}), \frac{1}{2}\right)$;
- is H-self-similar, that is, for any $a > 0$, $X(at) \overset{f.d.d.}{=} a^H X(t)$, where

$$H = kd - k + k/2 + 1 = kd - k/2 + 1 \in (1/2, 1);$$

- has stationary increments.

We say: The Hermite process is H-sssi.

Generalized Hermite kernel

In the definition of the Hermite process

$$Z(t) = \int_{\mathbb{R}^k}' \left\{ \int_0^t \prod_{j=1}^{k} (s - x_j)_+^{d-1} ds \right\} W(dx_1) \dots W(dx_k),$$

replace the product of functions in the bracket, by

$$g(s - x_1, \cdots, s - x_k) 1\{s\mathbf{1} > \mathbf{x}\}$$

where g is a multivariate homogeneous function.

• This idea goes back to Mori and Oodaira (1986) who use it to study the law of iterated logarithm.

Definition. We say that a nonzero measurable function $g(\mathbf{x})$ defined on $\mathbb{R}_+^k$ is a *generalized Hermite kernel*, if it satisfies

1. $g(\lambda \mathbf{x}) = \lambda^\alpha g(\mathbf{x})$, $\forall \lambda > 0$, where $\alpha \in (-\frac{k+1}{2}, -\frac{k}{2})$;
2. $\int_{\mathbb{R}_+^k} |g(\mathbf{x}) g(\mathbf{1} + \mathbf{x})| d\mathbf{x} < \infty$.

Definition. The process

$$Z(t) := \int_{\mathbb{R}^k}' \int_0^t g(s - x_1, \ldots, s - x_k) 1_{\{s > x_1, \ldots, s > x_k\}} ds \; W(dx_1) \ldots W(dx_k)$$

where g is a generalized Hermite kernel, is called a *generalized Hermite process*. It is well-defined in $L^2(\mathbb{R}^k)$, $\forall t > 0$, and it is H-sssi with

$$H = \alpha + k/2 + 1 \in (1/2, 1).$$

Example 1

Let

$$Z_{\gamma_1,\gamma_2}(t) = A \int_{\mathbb{R}^2}' \int_0^t [(s-x_1)_+^{\gamma_1}(s-x_2)_+^{\gamma_2}]ds \; W(dx_1)W(dx_2),$$

where $A \neq 0$ is a constant, $\gamma_1, \gamma_2 \in (-1, -1/2)$ and $\alpha = \gamma_1 + \gamma_2 > -3/2$. We shall call $Z_{\gamma_1,\gamma_2}(t)$ a *generalized Rosenblatt process*. It is an H-sssi process with

$$H = \alpha + k/2 + 1 = \alpha + 2 = \in (1/2, 1).$$

• We shall call $Z_{\gamma_1,\gamma_2}(t)$ a *generalized Rosenblatt process*.
Maejima and Tudor (2012) called it a *non-symmetric Rosenblatt process*.
• Here:

$$g(x_1, x_2) = x_1^{\gamma_1} x_2^{\gamma_2}.$$

• If $\gamma_1 = \gamma_2$, we get the usual Rosenblatt process.

Example 2

Let

$$g(x_1, \ldots, x_k) = \max\left(\frac{x_1 \ldots x_k}{x_1^{k-\alpha} + \ldots + x_k^{k-\alpha}},\ x_1^{\alpha/k} \ldots x_k^{\alpha/k}\right)$$

where $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$ and $\alpha \in (-k/2 - 1/2, -k/2)$.

- Note that g is homogeneous of degree α.

- The corresponding $H - SSSI$ generalized Hermite process with $H = \alpha + k/2 + 1$ is

$$\int'_{\mathbb{R}^k} \int_0^t \left(\frac{\prod_{j=1}^k (s - x_j)_+}{\sum_{j=1}^k (s - x_j)_+^{k-\alpha}}\right) \vee \left(\prod_{j=1}^k (s - x_j)_+^{\alpha/k}\right) ds \prod_{j=1}^k W(dx_j).$$

How to obtain an H-sssi process with $0 < H < 1/2$?

Theorem

The process $Z^\beta(t) =$

$$\int'_{\mathbb{R}^k}\int_{\mathbb{R}} \frac{1}{\beta}\Big[(t-s)_+^\beta - (-s)_+^\beta\Big] g(s-x_1,\dots,s-x_k) 1_{\{s>x_1,\dots,s>x_k\}} ds \prod_{j=1}^k W(dx_j),$$

is an H-sssi process with $H = \alpha + \beta + k/2 + 1 \in (0,1)$.

• g is homogeneous with exponent $\alpha \in (-k/2 - 1/2, -k/2)$ and

$$-1 < -\alpha - \frac{k}{2} - 1 < \beta < -\alpha - \frac{k}{2} < \frac{1}{2}.$$

These condition ensure that the integrand is in $L^2(\mathbb{R}^k)$.

• The case $\beta = 0$ leads back to the generalized Hermite process case where we have $1_{[0,t]}(s)$.

The spectral perspective

The L^2 Fourier transform of the integrand

$$h_t^\beta(\mathbf{x}) = \frac{1}{\beta}\left[(t-s)_+^\beta - (-s)_+^\beta\right] g(s-x_1,\dots,s-x_k) 1_{\{s>x_1,\dots,s>x_k\}} ds$$

is

$$\widehat{h}_t^\beta(\boldsymbol{\nu}) = \frac{(e^{it\langle \boldsymbol{\nu},\mathbf{1}\rangle}-1)}{i\langle \boldsymbol{\nu},\mathbf{1}\rangle)}(i\langle \boldsymbol{\nu},\mathbf{1}\rangle)^{-\beta}\widehat{g}(-\boldsymbol{\nu})\Gamma(\beta),\ a.e.,$$

where $\mathbf{x},\boldsymbol{\nu}\in\mathbb{R}^k$ and $\langle \boldsymbol{\nu},\mathbf{1}\rangle = \sum_{j=1}^k \nu_j$. Thus,

$$Z_H^{(k)}(t) = c\int_{\mathbb{R}^k}'' \frac{(e^{it\langle \boldsymbol{\nu},\mathbf{1}\rangle}-1)}{i\langle \boldsymbol{\nu},\mathbf{1}\rangle)}(i\langle \boldsymbol{\nu},\mathbf{1}\rangle)^{-\beta}\widehat{g}(-\boldsymbol{\nu})\widehat{W}(d\nu_1)\dots\widehat{W}(d\nu_k),$$

One obtains back the Hermite process by setting $\beta = 0$ and

$$\widehat{g}(-\boldsymbol{\nu}) = c\prod_{j=1}^k |\nu_k|^{-d}.$$

Multivariate limit theorems involving generalized Hermite processes

Polynomial forms

Consider the polynomial form of order k:

$$X(n) = \sum_{0<i_1,\dots,i_k<\infty}' a(i_1,\dots,i_k)\epsilon_{n-i_1}\dots\epsilon_{n-i_k},$$

where the prime $'$ indicates that one doesn't sum on the diagonals $i_p = i_q$ $p \neq q$ and where the $\sum_{\mathbf{i}>\mathbf{0}}' a(\mathbf{i})^2 < \infty$, so that $X(n)$ is well-defined in the $L^2(\Omega)$-sense. For simplicity, take

$$a(i_1,\dots,i_k) = g(i_1,\dots,i_k),$$

where g is a generalized Hermite kernel with some regularity conditions.

- Say that $X(n)$ is short-range dependent (SRD) if the sum of the covariances of $X(n)$ converges.
- Say that $X(n)$ is long-range dependent (LRD) if the sum of the covariances of $X(n)$ diverges.

Limit theorems

Consider first a single polynomial form of order k:

$$X(n) = \sum_{0<i_1,\ldots,i_k<\infty}' g(i_1,\ldots,i_k)\epsilon_{n-i_1}\ldots\epsilon_{n-i_k}.$$

We are interested in the limit of

$$\frac{1}{N^H}\sum_{n=1}^{[Nt]} X(n)$$

for special choices of the kernel $a(i_1,\ldots,i_k)$. The limit can be:

- BM if $k=1$ (linear process), SRD, $H=1/2$
- BM if $k\geq 2$, (linear or non-linear process), SRD, $H=1/2$
- $Z(t)$ if $k\geq 1$ (generalized Hermite), LRD, $H\in(1/2,1)$
- $Z^\beta(t)$ if $k\geq 1$ (fractionally filtered generalized Hermite), SRD or LRD, $H\in(0,1)$.

Consider a **vector** of such processes

The components are expressed as

$$\frac{1}{N^H}\sum_{n=1}^{[Nt]} X(n) = \frac{1}{N^H}\sum_{n=1}^{[Nt]} \sum_{0<i_1,\dots,i_k<\infty}^{\prime} g(i_1,\dots,i_k)\epsilon_{n-i_1}\dots\epsilon_{n-i_k}.$$

The components have

- the same iid noise $\{\epsilon_k, k \in \mathbb{Z}\}$
- different kernels $g(i_1,\dots,i_k)$
- different orders k
- different exponents H
- SRD, LRD, fLRD

Question: Is there joint convergence? **Answer:** Yes

Multivariate limit theorem

Theorem (Bai and Taqqu (2014))

The joint convergence holds with limit vector

$$\Big(\mathbf{W}(t), \mathbf{B}(t), \mathbf{Z}(t), \mathbf{Z}^{\boldsymbol{\beta}}(t)\Big),$$

where

(i) $\mathbf{W}(t) := (\sigma_1 W(t), \ldots, \sigma_{J_{S_1}} W(t))$ *defined by the same standard Brownian motion* $W(t)$. *[$k = 1$, SRD]*

(ii) $\mathbf{B}(t)$ *is a multivariate Brownian motion with joint covariance* 0 *if the orders* $k_p \neq k_q$. *[$k \geq 2$, SRD]*

(iii) $\mathbf{Z}(t)$ *is a multivariate generalized Hermite process and using the* $W(t)$ *in Point* (i) *as Brownian motion integrator.* *[$k \geq 1$, LRD]*

(iv) $\mathbf{Z}^{\beta}(t)$ *is a multivariate fractionally-filtered generalized Hermite process and using the* $W(t)$ *in Point* (i) *as Brownian motion integrator.* *[$k \geq 1$, SRD or LRD depending on β]*

Independence / dependence between the components

The limit vector is:

$$\Big(\mathbf{W}(t), \mathbf{B}(t), \mathbf{Z}(t), \mathbf{Z}^{\boldsymbol{\beta}}(t)\Big).$$

$\mathbf{B}(t)$ is always independent of $(\mathbf{W}(t), \mathbf{Z}(t), \mathbf{Z}^{\boldsymbol{\beta}}(t))$. Indeed:

• $\mathbf{B} \perp\!\!\!\perp \mathbf{W}$ [$k \geq 2$ versus $k = 1$]
• The processes $\mathbf{B}(t), \mathbf{Z}(t)$ and $\mathbf{Z}^{\boldsymbol{\beta}}(t)$ involve the same integrator $W(\cdot)$ because they are defined in terms of the same ϵ_i's.
• $(\mathbf{Z}, \mathbf{Z}^{\boldsymbol{\beta}})$ involve only $W \Longrightarrow \mathbf{B}_2 \perp\!\!\!\perp (\mathbf{Z}, \mathbf{Z}^{\boldsymbol{\beta}})$.

Dependence between LRD limit components

From Ustunel and Zakai (1989) , we have the following criterion for the independence of multiple Wiener-Itô integrals: suppose that symmetric $g_1 \in L^2(\mathbb{R}^p)$ and $g_2 \in L^2(\mathbb{R}^q)$, $(p, q \geq 1)$. Then the multiple integrals $I_p(g_1)$ and $I_q(g_2)$ which share the same random measure are independent if and only if

$$g_1 \otimes_1 g_2 := \int_{\mathbb{R}} g_1(x_1, \ldots, x_{p-1}, u) g_2(x_p, \ldots, x_{p+q-2}, u) du = 0 \quad a.s..$$

One can apply this to the Hermite processes:

$$\begin{aligned} &(g_{p,d} \otimes_1 g_{q,d})(x_1, \ldots, x_{p+q-2}) \\ = &\int_{\mathbb{R}} \left(\int_0^t \prod_{j=1}^{p-1} (s - x_j)_+^{d-1} (s-u)_+^{d-1} ds \int_0^t \prod_{j=p}^{p+q-2} (s-x_j)_+^{d-1} (s-u)_+^{d-1} ds \right) du > 0 \end{aligned}$$

since everything involved in the integrand is positive. So the Hermite processes of different orders are dependent.
In general?

Are the generalized Hermite processes more general than the Hermite processes?

Answer: YES

Recall the generalized Rosenblatt process:

$$Z_{\gamma_1,\gamma_2}(t) = A \int_{\mathbb{R}^2}' \int_0^t [(s-x_1)_+^{\gamma_1}(s-x_2)_+^{\gamma_2} ds \; B(dx_1)B(dx_2),$$

where $\gamma_1, \gamma_2 \in (-1, -1/2)$ and $\alpha = \gamma_1 + \gamma_2 > -3/2$.

Choose A so that it has variance 1.

The third cumulant

γ_1	-0.600	-0.589	-0.579	-0.568	-0.558	-0.547	-0.537	-0.526	-0.516	-0.505
$K_3(\gamma_1, \alpha - \gamma_1)$	2.548	2.549	2.554	2.559	2.564	2.561	2.538	2.465	2.258	1.587

Table: $K_3(\gamma_1, \alpha - \gamma_1)$ when $\alpha = -1.2$ (or $H = 0.8$).

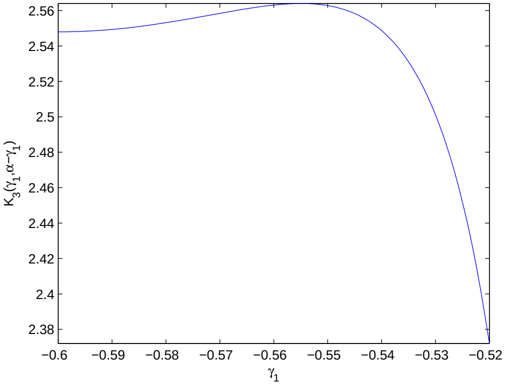

Figure: $K_3(\gamma_1, \alpha - \gamma_1)$ when $\alpha = -1.2$ (or $H = 0.8$).

Bibliography

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Thank you!