Rokhlin dimension for actions of residually finite groups

Dynamics and C*-Algebras: Amenability and Soficity
Banff International Research Station

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Within the theory of C^*-algebras, the crossed product construction is long known as a nearly inexhaustible source of interesting examples. In particular, certain crossed products are often natural test cases for the Elliott classification program. Motivated by the increasing importance of nuclear dimension in the Elliott program, the following question constitutes the main theme of this talk:

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Let A be a C^*-algebra, G a countable group and $(\alpha, w) : G \curvearrowright A$ a cocycle action. How does nuclear dimension behave with respect to passing to the (twisted) crossed product? $\quad A \rightsquigarrow A \rtimes_{\alpha,w} G$

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Answering this question in full generality seems to be far out of reach at the moment. However, by inventing the concept of Rokhlin dimension, Hirshberg, Winter and Zacharias have paved the way towards very satisfactory partial answers. This notion was initially defined for actions of finite groups and integers, and was also adapted to actions of $\mathbb{Z}^m$.

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We will discuss a generalization to cocycle actions of residually finite groups:

Definition

Let A be a separable C^*-algebra and G a countable, discrete group and $H \subset G$ a subgroup with finite index. Let $(\alpha, w) : G \curvearrowright A$ be a cocycle action. Let $d \in \mathbb{N}$ be a natural number.

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$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F_\infty(A), \alpha_\infty) \quad (l = 0, \dots, d)$$

with $\varphi_0(\mathbf{1}) + \dots + \varphi_d(\mathbf{1}) = \mathbf{1}$.
If no number satisfies this condition, we set $\dim_{\mathrm{Rok}}(\alpha, H) := \infty$.

Definition

Let A be a separable C^*-algebra and G a countable, discrete and residually finite group. Let $\sigma = (G_n)_n$ be a residually finite approximation of G, i.e. a decreasing and separating sequence of subgroups with finite index. We define

$$\dim_{\mathrm{Rok}}(\alpha, \sigma) = \sup_{n \in \mathbb{N}} \dim_{\mathrm{Rok}}(\alpha, G_n)$$

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Remark

If G is finite or $\mathbb{Z}^m$, the second expression agrees with the previously known definition.

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Theorem (Hirshberg-Winter-Zacharias)

If $\alpha : G \curvearrowright A$ is a finite group action on a unital C^*-algebra, we have

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If A *is a unital* C^**-algebra and* $\alpha \in \mathrm{Aut}(A)$*, we have*

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha \mathbb{Z}) \leq 2 \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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Theorem (S)

If $\alpha : \mathbb{Z}^m \curvearrowright A$ is an action on a unital C^*-algebra, we have

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha G) \leq 2^m \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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Theorem (S-Wu-Zacharias)

Let G be a countable, discrete, residually finite group. Let σ be a residually finite approximation of G. Let A be any C^*-algebra and $(\alpha, w) : G \curvearrowright A$ a cocycle action. Then we have

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\Box_\sigma G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha, \sigma) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

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The above constant denotes the asymptotic dimension of the so-called box space of G associated to σ. We shall elaborate in the next part.

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Remark

One particular instance of the above inequality is

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_{\alpha,w} G) \leq \mathrm{asdim}^{+1}(\Box_s G) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A),$$

where $\Box_s G$ is a so-called standard box space of G.

Definition (Roe, Khukhro)

Let G be a countable, discrete and residually finite group. Let $\sigma = (G_n)_n$ be a residually finite approximation of G. Equip G with a right-invariant, proper metric d. (e.g. right-invariant word-length metric.)

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Let G be a countable, discrete and residually finite group. Let $\sigma = (G_n)_n$ be a residually finite approximation of G. Equip G with a right-invariant, proper metric d. (e.g. right-invariant word-length metric.)
The box space $\Box_\sigma G$ of G associated to σ is the disjoint union $\bigsqcup_{n\in\mathbb{N}} G/G_n$, endowed with a metric d_B such that this metric, when restricted to some G/G_n, is induced by d under the quotient map $G \longrightarrow\!\!\!\!\!\rightarrow G/G_n$, and such that

$$\mathrm{dist}_{\mathrm{d}_B}(G/G_n, G/G_m) \geq \max\left\{\mathrm{diam}_{\mathrm{d}_B}(G/G_n), \mathrm{diam}_{\mathrm{d}_B}(G/G_m)\right\}$$

for all $n, m \in \mathbb{N}$.

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If σ is dominating, we will call $\Box_s G = \Box_\sigma G$ a standard box space.

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So for what kind of groups do we have $\operatorname{asdim}(\Box_s G) < \infty$?

Example

- The box space of a finite group is always coarsely equivalent to a one-point space, hence it has asymptotic dimension 0.
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Finitely generated, virtually nilpotent groups G satisfy $\mathrm{asdim}(\Box_s G) < \infty$.

Remark

This result has recently been slightly improved by Wu, in that

$$\mathrm{asdim}^{+1}(\Box_\sigma G) \leq 3^{\ell_{\mathrm{Hir}}(G)}$$

for every residually finite approximation σ of G.

When a box space $\Box_\sigma G$ of a residually finite group has finite asymptotic dimension, one might be tempted to think that this value encodes the complexity of the group G in some sense with respect to σ. This turns out to be true, in that the value simultaniously keeps track of both large-scale geometry and periodic behavior.

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Lemma (S-Wu-Zacharias)

Let G be a residually finite group and $\sigma = (G_n)_n$ a residually finite approximation. Then for every $s \in \mathbb{N}$, the statement $\operatorname{asdim}(\Box_\sigma G) \leq s$ is equivalent to the following condition:

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- $\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)})h = \emptyset$ *for all l and $h \in G_n \setminus \{1\}$.*
- $\sum_{l=0}^{s} \sum_{h \in G_n} \mu^{(l)}(gh) = 1$ *for all $g \in G$.*
- *Each $\mu^{(l)}$ is (M, ε)-flat with respect to left-translation, i.e. $\|\mu^{(l)} - \mu^{(l)}(g \cdot \_)\|_\infty \leq \varepsilon$ for all l and $g \in M$.*

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For all $(\alpha, w) : G \curvearrowright A$ *and* σ*, we have*

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Proof (rough sketch of main theorem for actions)

Let $F \Subset A \rtimes_{\alpha,w} G$ and $\varepsilon > 0$. We may pretend that F consists of au_g for certain $a \in A$ and $g \in M \Subset G$. Choose $n \in \mathbb{N}$ and decay functions $\mu^{(0)}, \dots, \mu^{(s)}$ according to the Lemma. Define $\mathcal{G}^{(l)} = \mathrm{supp}(\mu^{(l)})$. Choose Rokhlin towers $\varphi_0, \dots, \varphi_d : (\mathcal{C}(G/G_n), G\text{-shift}) \longrightarrow (F_\infty(A), \alpha_\infty)$.

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We thus obtain $\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha G) \leq (s+1)(d+1)(r+1)$.

Let G be a countable, discrete group and $d \in \mathbb{N}$. Let $\triangle_d G \subset \ell^1(G)$ be the set of all probability measures of G supported on at most $d+1$ points. Let $\triangle G = \bigcup_{d \in \mathbb{N}} \triangle_d G$ be the set of all finitely supported probability measures.

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Let $\alpha : G \curvearrowright X$ be an action on a compact metric space. Then α is amenable if there exist approximately equivariant maps

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That is, there exists a net of continuous maps $f_\lambda : X \to \triangle G$ such that $\|f_\lambda(\alpha_g(x)) - \beta_g(f_\lambda(x))\|_1 \to 0$ as $\lambda \to \infty$ for all $x \in X$ and $g \in G$.

Definition (Guentner-Willett-Yu)

Let $\alpha : G \curvearrowright X$ be an action on a compact metric space and $d \in \mathbb{N}$. α is said to have amenability dimension at most d, written $\dim_{\mathrm{am}}(\alpha) \leq d$, if there exist approximately equivariant maps

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Theorem (Guentner-Willett-Yu)

For a free action $\alpha : G \curvearrowright X$, one has the estimate

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{C}(X) \rtimes_\alpha G) \leq \dim_{\mathrm{am}}^{+1}(\alpha) \cdot \dim^{+1}(X).$$

For a topological dynamical system (X, α, G), denote by $\bar{\alpha} : G \curvearrowright \mathcal{C}(X)$ the induced C^*-algebraic action.

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This can be answered as follows:

Theorem (S-Wu-Zacharias)

Let σ be a residually finite approximation of G. If $\alpha : G \curvearrowright X$ is free, one has the following estimates:

$$\dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}) \;\leq\; \dim_{\mathrm{am}}^{+1}(\alpha) \;\leq\; \mathrm{asdim}^{+1}(\Box_\sigma G) \cdot \dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}, \sigma).$$

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In particular, if $\mathrm{asdim}(\Box_s G) < \infty$, then α has finite amenability dimension if and only if $\bar{\alpha}$ has finite Rokhlin dimension.

Last year, the following result was obtained:

Theorem (S)

If $\alpha : \mathbb{Z}^m \curvearrowright X$ *is a free action on a compact metric space of finite covering dimension, then* $\dim_{\mathrm{Rok}}(\bar{\alpha}) < \infty$*. Hence* $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m) < \infty$*.*

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Combining

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- detailed study of amenability dimension
- some geometric group theory involving nilpotent groups

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Combining

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this extends to the following setting:

Theorem (S-Wu-Zacharias)

Let G be a finitely generated, nilpotent group. If $\alpha : G \curvearrowright X$ is a free action on a compact metric space of finite covering dimension, then both $\dim_{\mathrm{am}}(\alpha)$ and $\dim_{\mathrm{Rok}}(\bar{\alpha})$ are finite. In particular, $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha G)$ has finite nuclear dimension. (In fact at most $3^{\ell_{\mathrm{Hir}}(G)} \cdot \dim^{+1}(X)^2 - 1$.)

Thank you for your attention!