A Spiky Ball

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Special Case: The Illumination Problem

Gohberg–Markus–Levi–Boltyanskii–Hadwiger Illumination Conjecture

Fix n. Then the maximum of $N(K, \text{int}\, K)$ over all convex bodies K in $\mathbb{R}^n$ is 2^n, and only attained by parallelotopes.

Known:
If K is smooth then $i(K) = n + 1$.

Rogers

$$i(K) := N(K, \text{int}\, K) \leq \begin{cases} 2^n(n \ln n + n \ln\ln n + 5n) & \text{if } K = -K, \\ \binom{2n}{n}(n \ln n + n \ln\ln n + 5n) & \text{otherwise.} \end{cases}$$

Main Result

Theorem

Let $1 < D < 1.116$ be given. Then for any, sufficiently large dimension n, there is an o-symmetric convex body K in $\mathbb{R}^n$, with illumination number

$$\mathrm{i}(K) = N(K, \operatorname{int} K) \geq .05D^n, \tag{1}$$

for which

$$\frac{1}{D}\mathbf{B}^n \subset K \subset \mathbf{B}^n. \tag{2}$$

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Sharp:

If $\frac{1}{D}\mathbf{B}^n \subset K \subset \mathbf{B}^n$ for some $D > 1$, then

$$\mathrm{i}(K) \leq \frac{n \ln n + n \ln \ln n + 5n}{\Omega_{n-1}(\alpha)}, \tag{3}$$

where $\alpha = \arcsin(1/D)$.

Application: Gap between ill and vein

$K = -K$

Illumination parameter [K. Bezdek '06]:

$$\mathrm{ill}(K) = \inf \left\{ \sum_{p \in \mathrm{vert}\, P} \|p\|_K \mid P \text{ a polytope such that } \mathrm{vert}\, P \text{ illuminates } K \right\}.$$

Vertex index [K. Bezdek – A. Litvak '07]:

$$\mathrm{vein}(K) = \inf \left\{ \sum_{p \in \mathrm{vert}\, P} \|p\|_K \mid P \text{ a polytope such that } K \subseteq P \right\}.$$

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Let K be a spiky ball. Then
$\mathrm{vein}(K)$ is of order $n^{3/2}$,
$\mathrm{ill}(K) \geq \mathrm{i}(K)$ is exponentially large.

Preliminaries

$u \in \mathbb{S}^{n-1}$, and $0 < \varphi < \pi/2$.

Spherical cap: $C(u,\varphi) = \{v \in \mathbb{S}^{n-1} \ : \ \sphericalangle(u,v) \leq \varphi\}$.

Probability measure of cap: $\Omega_{n-1}(\varphi)$.

Lemma (Böröczky – Wintsche '03)

Let $0 < \varphi < \pi/2$. Then

$$\Omega_n(\varphi) > \frac{\sin^n \varphi}{\sqrt{2\pi(n+1)}}, \tag{4}$$

$$\Omega_n(\varphi) < \frac{\sin^n \varphi}{\sqrt{2\pi n}\cos\varphi}, \qquad \text{if } \varphi \leq \arccos\frac{1}{\sqrt{n+1}}, \tag{5}$$

$$\Omega_n(t\varphi) < t^n \Omega_n(\varphi), \qquad \text{if } 1 < t < \frac{\pi}{2\varphi}. \tag{6}$$

Roughly,

$$\Omega_n(\varphi) \approx \sin^n \varphi.$$

The construction

$X_1, \ldots, X_N$ independent random points on $\mathbb{S}^n$.

$$K = \operatorname{conv}\left(\{\pm X_i \ : \ i \in [N]\} \cup \frac{1}{D}\mathbf{B}^{n+1} \right).$$

Clearly, K is o-symmetric and $\frac{1}{D}\mathbf{B}^{n+1} \subset K \subset \mathbf{B}^{n+1}$.
Need: illumination number not small.

Bad event E_1

Notation: $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ is such that $\sin\alpha = 1/D$.

E_1

The event that there are $i \neq j \in [N]$ with $\sphericalangle(X_i, X_j) < \pi - 2\alpha$ or $\sphericalangle(-X_i, X_j) < \pi - 2\alpha$.

If E_1 does not occur, then

for all $i \in [N]$:
the set of directions that illuminate K at X_i is the spherical cap $C(-X, \alpha)$.

Bad event E_2

$T \in \mathbb{Z}^+$ fixed.

E_2

There is a direction $u \in \mathbb{S}^n$ with $|C(u, \alpha) \cap \{\pm X_i \ : \ i \in [N]\}| > T$.

If $NOT(E_1)$ AND $NOT(E_2)$, then

$$\mathrm{i}(K) \geq 2N/T.$$

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Problem

$\mathbb{P}(E_2)$ is hard to estimate.

E_2': finitize E_2

E_2

There is a direction $u \in \mathbb{S}^n$ with $|C(u,\alpha) \cap \{\pm X_i \ : \ i \in [N]\}| > T$.

Fix $\delta > 0$.
Let Λ be a δ-net of $\mathbb{S}^n$.
Let $p = 2\Omega_n(\alpha+\delta)$.
Let $\Theta > 1$ be fixed, and set $T = N\Theta p$.

E_2'

There is a direction $v \in \Lambda$ with $|C(v,\alpha+\delta) \cap \{\pm X_i \ : \ i \in [N]\}| > N\Theta p$.

Clearly, if $E_2 \implies E_2'$.

If $NOT(E_1)$ AND $NOT(E_2')$, then

$$\mathrm{i}(K) \geq 2/(\Theta p).$$

Is it possible?

If $NOT(E_1)$ AND $NOT(E_2')$, then

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The task

Need to set N, Θ, δ such that $\mathbb{P}(\text{not}(E_1) \text{ and not}(E_2')) > 0$ and $2/(\Theta p)$ is exponentially large in n.

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Equation 1

$$\mathbb{P}(E_1) \leq N^2 \Omega_n(\pi - 2\alpha) \leq 1/4.$$

How to set N, Θ, δ

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$$\mathbb{P}(E_2') \leq |\Lambda| \, \mathbb{P}(\xi > N\Theta p) \leq 1/4 \quad \text{with } \xi \sim \texttt{Binom}(N, p).$$

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Easy: There is a Λ with $|\Lambda| \leq n^2 / \sin^n(\delta)$.

Equation 2

$$\mathbb{P}(E_2') \leq \frac{n^2}{\sin^n(\delta)} \mathbb{P}(\xi > N\Theta p) \leq 1/4 \quad \text{with } \xi \sim \texttt{Binom}(N, p).$$

Can we set N, Θ, δ properly?

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$$\mathbb{P}(E_1) \leq N^2 \Omega_n(\pi - 2\alpha) \leq 1/4.$$

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Equation 3

Make $2/(\Theta p)$ exponentially large.

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Yes, we can!

Good event: Happy Birthday, Karcsi and Egon!