Double-normal pairs in a finite set of points

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Joint work with János Pach

Overview

Introduction

The problem statement

The plane

Points on a 2-sphere

3-space

Higher dimensions

Overview

Extremal problems in combinatorial geometry

V a set of n points in $\mathbb{R}^d$.

Join pairs (or triples, etc.) satisfying certain properties, e.g.

1. unit distance pairs (Erdös 1946)
2. unit area triangles (Oppenheim 1967)
3. diameter pairs (Hopf–Pannwitz 1934)
4. antipodal pairs (Klee 1960)
5. double-normal pairs (Martini–Soltan 2005)

to define a graph (or hypergraph) on V.

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General Extremal Problem

Determine the maximum number of edges in such a graph (hypergraph) defined on a set of n points in $\mathbb{R}^d$.

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Double-normal pairs

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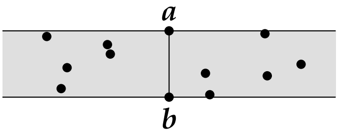

A double-normal pair is a pair $\boldsymbol{a}, \boldsymbol{b} \in V$ such that V is inside the strip bounded by the hyperplanes parallel through $\boldsymbol{a}$ and $\boldsymbol{b}$ perpendicular to $\boldsymbol{ab}$.

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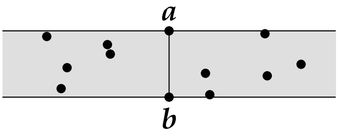

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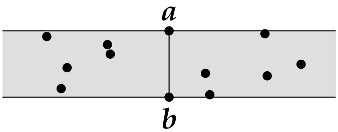

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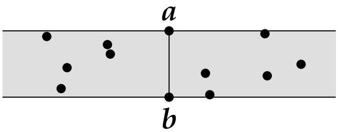

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Problem (Martini-Soltan 2005)

Determine the maximum number of double-normal pairs $N_d(n)$ among n points in $\mathbb{R}^d$.

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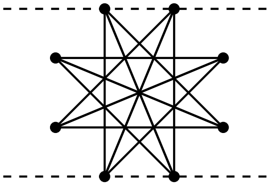
If n is even, $N_2(n) \geq \frac{3n}{2}$.



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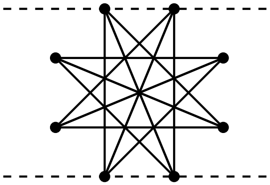

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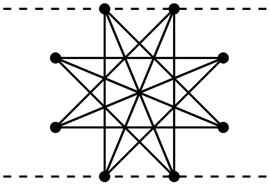

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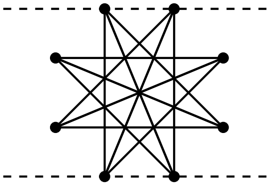

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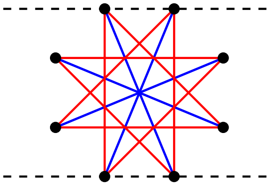

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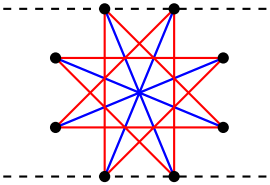

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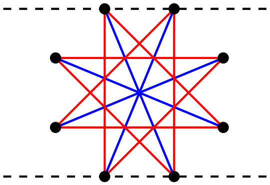

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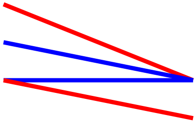

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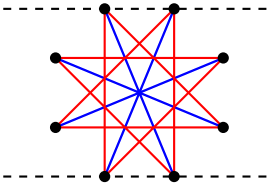

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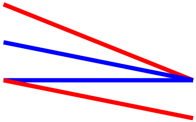

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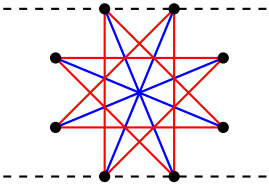

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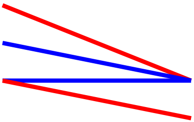

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The odd case needs some more analysis. ■

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Suppose first for simplicity that V is a set of n points on the 2-sphere in $\mathbb{R}^3$.

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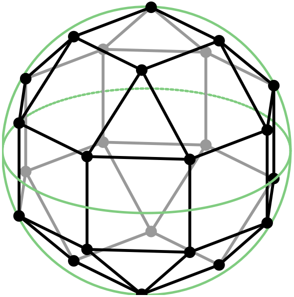

$n = 24$
small rhombicuboctahedron

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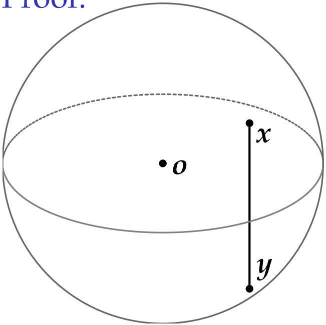

Consider double-normal pair $\boldsymbol{xy}$.

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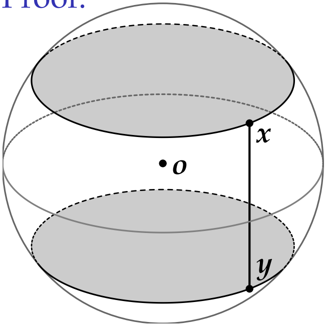

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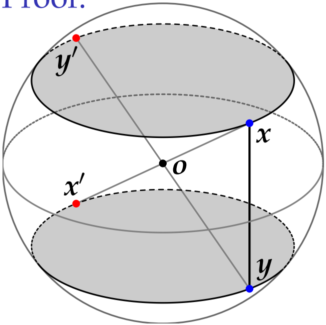

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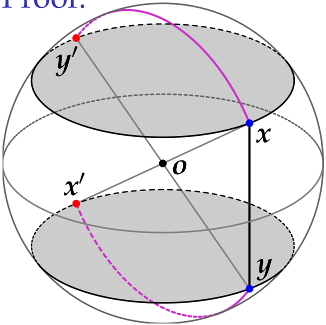

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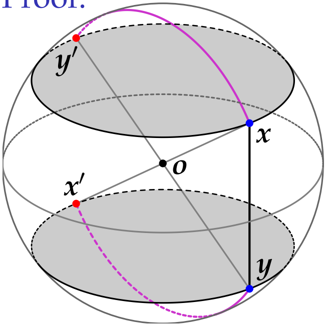

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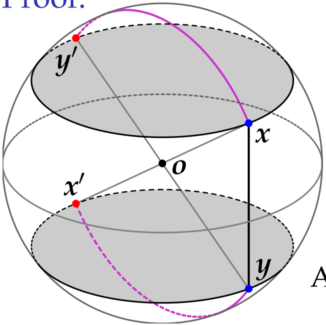

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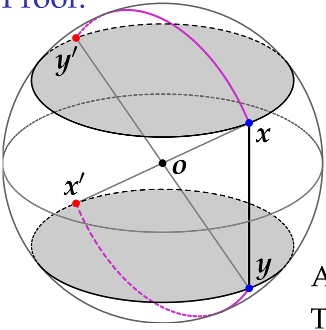

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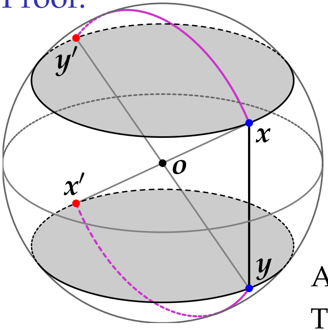

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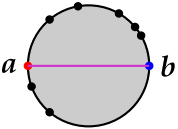
G_1 and G_2 are weak Gabriel graphs:
For each edge $\boldsymbol{ab}$*, there is no vertex inside the circular cap with diameter* $\boldsymbol{ab}$.



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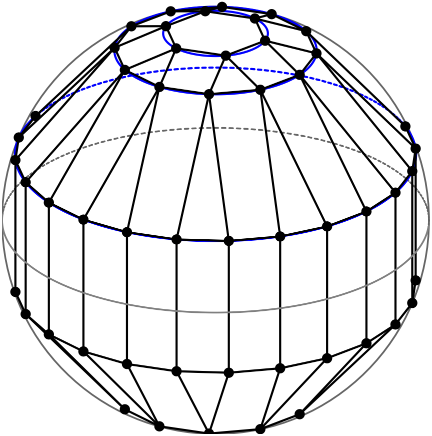

■

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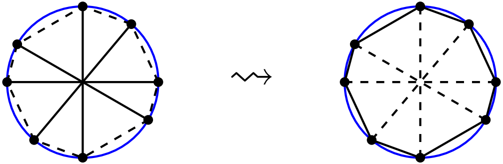
Replace a maximal collection of crossing arcs by a polygon:



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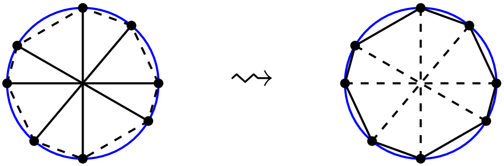

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Finishing the proof

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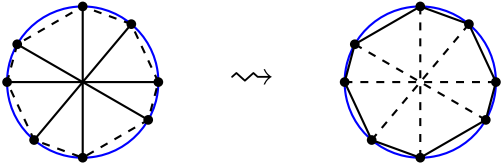

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Let g_i be the number of new regions bounded by i edges.

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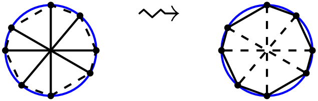

Therefore, each vertex is incident to at most 3 new regions.

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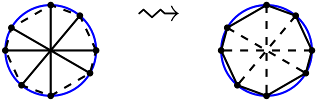

Therefore, each vertex is incident to at most 3 new regions. Double-count (vertex, new region)-pairs:

$$4g_4 + 6g_6 + \cdots \leq 3|V| \implies g_4 \leq \frac{3}{4}|V| .$$

Finishing the proof

We obtained four inequalities:

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So the number of original edges is at most

$$
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|E| &\overset{1.}{\leq} \left|E'\right| + 2g_4 + 3g_6 + \dots \\
&\overset{2.}{\leq} 3|V| - 6 - f_4 + 2g_4 - 2f_5 - 3f_6 + 3g_6 - 4f_7 - 5f_8 + 4g_8 - \cdots \\
&\overset{3.}{\leq} 3|V| - 6 + g_4 \\
&\overset{4.}{\leq} \frac{15}{4}|V| - 6 \qquad \blacksquare
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Overview

Double-normal pairs in $\mathbb{R}^3$

Theorem (Pach-S 2014+)

$$N_3(n) = \frac{n^2}{4} + o(n^2).$$

Proof.

Lower bound: carefully choose points on two perpendicular, skew circular arcs.

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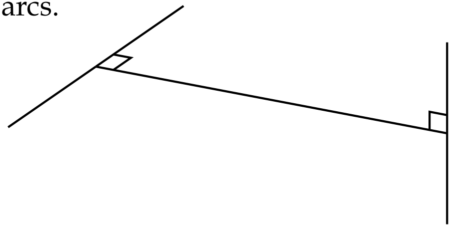

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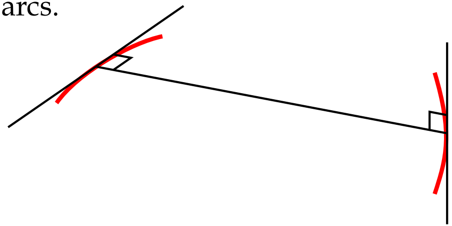

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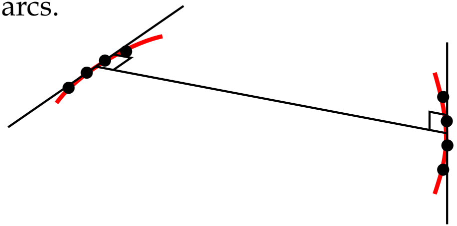

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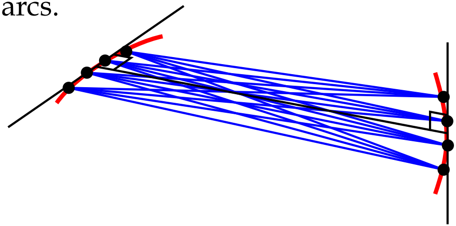

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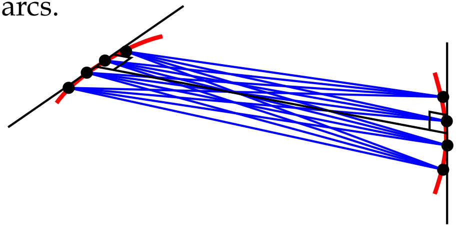

Upper bound: Ramsey argument with geometry to show that there is no $K_{m,m,m}$ for m large. Then apply Erdös–Stone.

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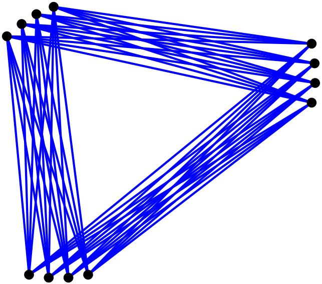

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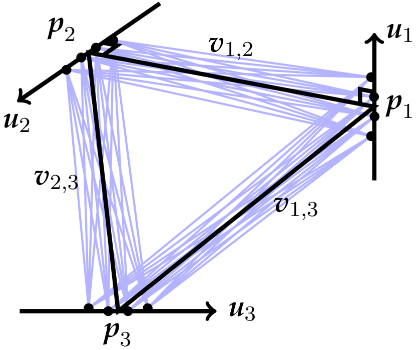

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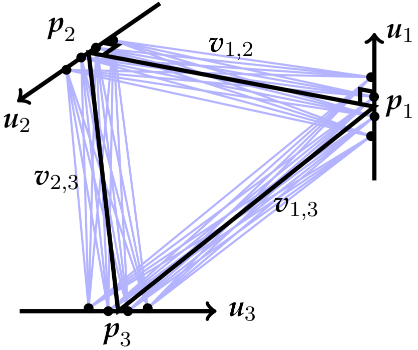

Points $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$

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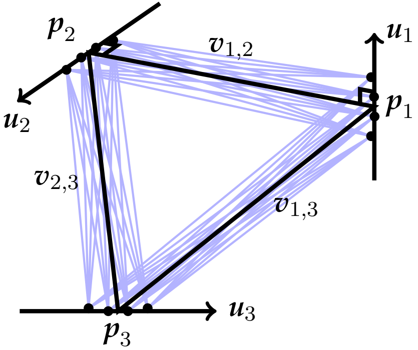

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Not all $\boldsymbol{p}_i$ are distinct. ■

Asymptotics of dense graphs

Observation

Given a collection $\mathcal{C}$ of graphs that are closed under induced subgraphs, let $f(n) = \max\{|E| \;:\; G = (V, E) \in \mathcal{C}, |V| = n\}$.

Then $f(n) = \frac{1}{2}\left(1 - \frac{1}{k_{\mathcal{C}}}\right)n^2 + o(n^2)$ *for some* $k_{\mathcal{C}} \geq 1$.

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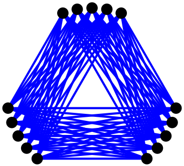
E.g. forbid $K_{m,m,m}$



to obtain an upper bound of $\frac{1}{4}n^2 + o(n^2)$ edges.

Overview

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Corollary

For each $d \geq 1$ there exists $k(d) \geq 1$ such that the number of double-normal pairs satisfies

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Thank you for your attention.

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