Karoly Bezdek and Egon Schulte are two geometers who have made important and numerous contributions to Discrete Geometry. Both have trained many young mathematicians and have actively encouraged the development of the area by professors of various countries.

Karoly Bezdek studied the illumination problem among many other problems about packing and covering of convex bodies. More recently he has also contributed to the theory of packing of equal balls in three-space and to the study of billiards, both of which are important for physicists. He has contributed two monographs [1, 2] in Discrete Geometry.

Egon Schulte, together with Ludwig Danzer, initiated the study on abstract polytopes as a way to enclose in the same theory objects with combinatorial properties resembling those of convex polytopes. An important cornerstone on this topic is his monograph with Peter McMullen [11]. In the latest years he explored the link between abstract polytopes and crystallography.

In this Workshop, which is a continuation of the preceding five-day one, we explored the ways in which the areas of abstract polytopes and discrete convex geometry has been influenced by the work of the two researchers. The last three decades have witnessed the revival of interest in these subjects and great progress has been made on many fundamental problems.

1 Abstract polytopes

1.1 Highly symmetric combinatorial structures in 3-space

Asia Ivić Weiss kicked off the workshop, discussing joint work with Daniel Pellicer [15] on the combinatorial structure of Schulte’s chiral polyhedra. An abstract polyhedron is a partially ordered set that mimics the face lattice of convex polyhedra. It is regular whenever its automorphism group (as partially ordered set) acts transitively on the flags (maximal chains), and it is chiral whenever its automorphism group induces two orbits on flags with the additional property that adjacent flags are in distinct orbits. The original motivation of this work was to determine the existence or non-existence of infinite abstract chiral polyhedra.

No finite abstract chiral polyhedron can be geometrically realized in 3-space with automorphisms corresponding to geometric symmetries. Egon Schulte found all (infinite) polyhedra in 3-space (that is, symmetric embeddings of abstract polyhedra) that are geometrically chiral. There are three infinite families of chiral polyhedra with finite faces, denoted by \( P(a, b), Q(c, d) \) and \( Q_1(c, d) \), and three infinite families with infinite faces, denoted by \( P_1(a, b), P_2(a, b), P_3(a, b) \). The question now becomes whether any of these is abstractly chiral.
Weiss and Pellicer proved that every polyhedron with finite faces that is geometrically chiral is also combinatorially chiral, hence providing an affirmative answer to the original question. On the other hand, they showed that the polyhedra with infinite faces are all abstractly regular, and that all chiral members in any of these three families are combinatorially isomorphic.

Undine Leopold talked about vertex-transitive polyhedra, where a polyhedron is here meant to be a closed, connected, orientable surface embedded in 3-space which is face-to-face tiled by finitely many plane, simple polygons (that are allowed to be non-convex and coplanar). Leopold has recently completed the classification of vertex-transitive polyhedra for tetrahedral rotation symmetry in her thesis [8], with no new examples besides the previously known polyhedron of genus 3.

For octahedral rotation symmetry there are three known examples, with the combinatorially regular Grünbaum polyhedron of genus 5 among them [6]. It is shown that the genus of any candidate map to be realizable as a polyhedron with this particular symmetry must belong to a finite set, bounded above by 31. However, several hundred polyhedral maps with a vertex-transitive action of the octahedral rotation group can be enumerated on surfaces of genus up to 31. Leopold presented an overview of the recent progress she made on closing this gap.

1.2 Constructions of highly symmetric abstract polytopes

Mark Mixer discussed recent progress on the problem of finding abstract regular polytopes of high rank with respect to the degree of their automorphism groups as transitive permutation groups. This is product of joint work with Peter Cameron, Maria Elisa Fernandes and Dimitri Leemans [4].

String C-groups of rank \(r\) are groups generated by \(r\) involutions satisfying some commutativity property and an intersection condition that hold for Coxeter groups with string diagram. They are in a one-to-one correspondence with the automorphism groups of regular \(r\)-polytopes [11, Section 2E]. Every string C-group can be represented by permutation graphs, where the number of vertices corresponds to the degree of the group. These graphs are used to show that if the group is neither alternating nor symmetric, and the degree of the group is \(n\) then the rank of the C-group (and of the polytope) must be at most \(n/2 + 1\) except for finitely many exceptions, which are completely classified.

Daniel Pellicer gave a new, purely combinatorial construction of abstract polyhedra from abstract 4-polytopes. This generalizes a geometric construction that has been previously used to describe Petrie’s skew regular polyhedron with square faces in 3-space as well as two of Coxeter’s regular skew polyhedra in 4-space [5]. It was shown that the automorphism group of a polyhedron constructed in this way is the extended automorphism group (the group of automorphisms and dualities) of the initial 4-polytope. Some examples of regular and chiral polyhedra obtained in this way were shown.

2 Discrete convex geometry

2.1 The structure of polytopes

Ted Bisztriczky gave a talk on recent progress that he has made on the combinatorial structure of neighbourly 4-polytopes. A neighbourly \(d\)-polytope is a polytope in which any \(k\) vertices form a face, for all \(k\) up to \(d/2\). Thus, in the 4-dimensional case the graph of the polytope is complete. It is known that there is, up to combinatorial equivalence, a unique neighbourly 4-polytope with 6 or 7 vertices, exactly 3 with 8 vertices, 23 with 9 vertices, and in the range from 333 to 432 on 10 vertices. Shemer [16] introduced a sewing construction that enabled the construction of many neighbourly polytopes. Bisztriczky introduced a generalization of the class of totally sewn neighborly polytopes, called \(k\)-linked polytopes, and studied their structure.

Nicholas Matteo, in his talk on four-orbit convex polytopes, presented a classification of all the convex polytopes with four flag orbits. They exist only in dimension 7 or less.

The polytopes with one flag orbit are of course the regular polytopes. The polytopes with two flag orbits are as close to being regular as possible, without being regular. These exist only in two or three dimensions, and are the cuboctahedron, the icosidodecahedron, and their duals, together with two infinite families of
2.2 Tilings and quasicrystals

Muhammad Khan presented his findings on the enumeration of polyomino tilings. He introduced two hypergraph polynomials, the edge cover polynomial and the edge decomposition polynomial. There is a natural way of associating a hypergraph to a tiling problem and he showed that the edge decomposition polynomial of this associated hypergraph is the generating function for the tilings. In this way edge decomposition polynomial can be used to count the number of tilings of a rectangular region by any finite set of polyominoes. He also gave a deletion-based recursive procedure for calculating the edge cover and edge decomposition polynomials of a given hypergraph.

Marjorie Senechal talked about tiling models of matter starting with the ancient Greeks. In the early 19th century tiling models became the paradigm for both crystal form and growth. The Crystallographic Restriction Theorem, stating that a rotation about a point of a 2- or 3-dimensional lattice is either 2-, 3-, 4-, or 6-fold, seemed to "outlaw" symmetry of order 5 in crystals.

Around 1974, Penrose (foreshadowed by Kepler) found tilings with no lattice structure, although having 5-fold symmetry in a generalized sense. His tilings were subsequently explained in a simpler way by De Bruijn as sections of a 5-dimensional lattice. Alan Mackay predicted the existence of quasicrystals with 5-fold symmetry in 1981. Dan Schechtman and his coworkers discovered quasicrystals in 1984. For this Schechtman received the Nobel prize in Chemistry in 2011. What was originally thought of as a minor upheaval in crystallography became a major shift. It might be that quasicrystals are a type of 3-dimensional Penrose tiling, but this is a problematic model, requiring too many tile shapes and too complicated matching rules. Also, the only quasicrystal so far whose structure has been determined (by Takakura et al. 2007), does not behave as a tiling. This creates doubts on the validity of the crystallographic restriction, opening several interesting problems for mathematicians and crystallographers.

2.3 Packing and covering by convex bodies

Alexander Litvak gave a talk on his work with Karoly Bezdek on covering and packing convex bodies by cylinders, following up from [3]. This work contributes to the generalizations and extensions of the celebrated plank problem of Tarski, originally solved by Bang, and later extended by Keith Ball and others. Here a question posed by Bang is investigated.

For $1 \leq k \leq d-1$, define a $k$-codimensional cylinder $C$ in $\mathbb{R}^d$ to be a set of the form $C = H + B$, where $H$ is a $k$-dimensional linear subspace of $\mathbb{R}^d$ and $B$ a measurable subset of $H^\perp$. Define the cross-sectional volume of $C$ with respect to a convex body $K$ by $\text{crv}_K(C) = |B|/|PK|$, where $P$ is the orthogonal projection onto $H^\perp$. This quantity is affine invariant.

It is shown that if $K$ is an ellipsoid and is covered by 1-codimensional cylinders $C_1, \ldots, C_N$, then $\sum_{i=1}^N \text{crv}_K(C_i) \geq 1$. It then follows by the Rogers-Shepard Theorem that if $K$ is an arbitrary convex body covered by 1-codimensional cylinders $C_1, \ldots, C_N$, then $\sum_{i=1}^N \text{crv}_K(C_i) \geq \frac{(d)}{k}^{-1}$. In the case $k = d-1$, Keith Ball proved the best possible lower bound of 1 in case $K$ is centrally symmetric. When $d = 3$ and $k = 1$, Bang conjectured that the lower bound should be 1/2. The above result gives 1/3.

The $k$-codimensional cylinders $C_i = B_i + H_i$ are defined to form a packing of $K$ if $B_i$ is contained in the orthogonal projection of $K$ onto $H_i^\perp$ and if $C_i \cap C_j \cap K = \emptyset$ for any distinct $i$ and $j$. It is shown that if $K$ is an ellipsoid in $\mathbb{R}^d$ and $C_1, \ldots, C_N$ are 1- or 2-codimensional cylinders that form a packing of $K$, then $\sum_{i=1}^N \text{crv}_K(C_i) \leq 1$.

Marton Naszódi talked about the classical illumination problem from convex geometry, due to Hadwiger and others. It is easy to see that the $d$-dimensional Euclidean ball can be covered by $d + 1$ translates of its interior. On the other hand, the $d$-dimensional cube needs $2^d$ translates of its interior, since no translate of the interior can cover more than one vertex. The Hadwiger conjecture states that each convex body can be covered by at most $2^d$ translates of its interior. This difficult conjecture is open already in dimension 3.

Naszódi shows that arbitrarily close to the Euclidean ball there are bodies needing a number of translates that is exponential in $d$. His example is a probabilistic construction [12].

polygons [9]. The polytopes with three flag orbits have also recently been classified by the speaker [10] (and exist only in dimensions 8 or less).
2.4 Combinatorial geometry

Konrad Swanepoel talked about his work with János Pach [13, 14] on a problem asked by Martini and Soltan [11]. Given a finite set $S$ of $n$ points in Euclidean space $\mathbb{R}^d$, we say that two points $a$ and $b$ from $S$ form a double-normal pair if $S$ lies in the closed slab bounded by the two hyperplanes orthogonal to the segment $ab$ passing through $a$ and $b$. Martini and Soltan asked for the determination of the largest number of double-normal pairs that can occur in a set of $n$ points in $\mathbb{R}^d$. In dimension 2 this maximum equals $3\lfloor n/2 \rfloor$. In dimension 3 there is a construction that places half of the points on a circular arc, and the other two points on another circular arc, in such a way to create a complete bipartite graph, resulting in $\lfloor n^2/4 \rfloor$ double-normal pairs. This turns out to be asymptotically sharp. For points in $\mathbb{R}^3$ that lie on a sphere there is an upper bound of $17n/4 - 6$, which is sharp for infinitely many $n$. In higher dimensions we show that asymptotically, the maximum number of double-normal pairs is $\frac{1}{2}(1 - 1/k(d))n^2 + o(n^2)$, where $d - O(\log d) \leq k(d) \leq d - 1$.

This work has recently been greatly improved by Andrei Kupavskii [7].

References