Property T for locally compact quantum groups

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(joint works with Prof. Chi-Keung Ng)

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• D. Kyed and M. Soltan partially generalize some equivalences of property T to discrete quantum groups.
• M. Daws, P. Fima, A. Skalski and S. White gave the definition of property T for general locally compact quantum groups, in their paper on the Haagerup property.

Preliminary

Notations and preliminaries on property T:

• If G is a locally compact group and $\mu : G \to \mathcal{L}(\mathfrak{H})$ is a continuous unitary representation, then a net $\{\xi_i\}_{i \in I}$ of unit vectors in $\mathfrak{H}$ is called an *almost invariant unit vector (for μ)* if

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(T2) $\exists$ minimal projection $p \in M(C^*(G))$ such that $\pi_{\mathbf{1}_G}(p) = 1$.

(T3) $\mathbf{1}_G$ is an isolated point in $\widehat{G}$.

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• The equivalences of property T with (T1)-(T5) for discrete quantum groups are some of the main results in the paper by Kyed and Soltan.

Property T for locally compact quantum groups

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- If $\mathbb{G}$ is a locally compact group G, then $\Delta(f)(s,t) = f(st)$; $\varphi(f) = \int f(t)\, d\lambda(t)$ (for $f \in L^1(G) \cap C_0(G)$); ψ is induced by the right Haar measure.

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- Unitary corepresentation: is a unitary $U \in M(\mathcal{K}(\mathfrak{H}) \otimes C_0(\mathbb{G}))$ satisfying $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$, where $\mathcal{K}(\mathfrak{H})$ is the algebra of all compact operators on $\mathfrak{H}$.

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- $(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}), \widehat{\Delta}^{\mathrm{u}})$: the universal C^*-bialgebra associated with the dual group $\widehat{\mathbb{G}}$ of $\mathbb{G}$.
- $V_{\mathbb{G}}^{\mathrm{u}} \in M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$: is a unitary that implements the bijective correspondence $U \longleftrightarrow \pi_U$ between the set of all unitary corepresentations of $C_0(\mathbb{G})$ and the set of all non-degenerate *-representations of $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ through the relation $U = (\pi_U \otimes \mathrm{id})(V_{\mathbb{G}}^{\mathrm{u}})$.

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— μ is *contained* in ν, denoted by $\mu \subset \nu$, if there exists an isometry $V : \mathfrak{H} \to \mathfrak{K}$ s.t. $\mu(a) = V^* \nu(a) V, \ \forall a \in A$;
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• $\widehat{A} \subseteq \mathrm{Rep}(A)$: the set of unitarily equiv. classes of all irreducible representations of A (equipped with Fell topology).

Property T for locally compact quantum groups

Lemma

Let A be a C^*-algebra and $(\mu, \mathfrak{H}) \in \widehat{A}$.
(a) The following statements are equivalent.

1. μ *is an isolated point in* $\widehat{A}$.
2. $\forall (\pi, \mathfrak{K}) \in \mathrm{Rep}(A)$ *with* $\mu \prec \pi$, *one has* $\mu \subset \pi$.
3. $A = \ker \mu \oplus \bigcap_{\nu \in \widehat{A} \setminus \{\mu\}} \ker \nu$.

(b) If $\dim \mathfrak{H} < \infty$, *then* $\{\mu\}$ *is a closed subset of* $\widehat{A}$.

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- For any unitary corepresentation $U \in M(\mathcal{K}(\mathfrak{H}) \otimes C_0(\mathbb{G}))$,

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- U has a non-zero invariant vector $\Leftrightarrow$ $\pi_{\mathbf{1}_\mathbb{G}} \subset \pi_U$;
- U has almost invariant vectors $\Leftrightarrow$ $\pi_{\mathbf{1}_\mathbb{G}} \prec \pi_U$.

Definition

$\mathbb{G}$ is said to have property T if every unitary corepresentation of $C_0(\mathbb{G})$ having an almost invariant unit vector has a non-zero invariant vector.

Property T for locally compact quantum groups

The following extends some results in the paper by Kyed and Soltan:

Proposition

Property T of a L.C.Q.G. $\mathbb{G}$ is equivalent to.

(1) $C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \cong \ker \pi_{\mathbf{1}_{\mathbb{G}}} \oplus \mathbb{C}$.

(2) $\exists$ *a projection* $p_{\mathbb{G}} \in M(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ *with* $p_{\mathbb{G}} C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) p_{\mathbb{G}} = \mathbb{C} p_{\mathbb{G}}$ *and* $\pi_{\mathbf{1}_{\mathbb{G}}}(p_{\mathbb{G}}) = 1$.

(3) $\pi_{\mathbf{1}_{\mathbb{G}}}$ *is an isolated point in* $\widehat{C_0^{\mathrm{u}}(\widehat{\mathbb{G}})}$.

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Recall:

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- Property T of G is equivalent to:

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For the corresponding results of (T4) and (T5), we need the notion of "tensor products" and "contragredients" of unitary corepresentations.

Property T for locally compact quantum groups

• If V is another unitary corepresentation of $C_0(\mathbb{G})$ on a Hilbert space $\mathfrak{K}$, then $U \textcircled{\top} V := U_{13}V_{23} \in M\big(\mathcal{K}(\mathfrak{H} \otimes \mathfrak{K}) \otimes C_0(\mathbb{G})\big)$ is a unitary corepresentation called the *tensor product* of U and V.

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• Recall that $\mathbb{G}$ is of *Kac type* if its antipode is bounded, or equivalently, $\widehat{\mathbb{G}}$ is *unimodular*, that is, the left and right Haar weights of $\widehat{\mathbb{G}}$ coincide.

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- Recall that $\mathbb{G}$ is of *Kac type* if its antipode is bounded, or equivalently, $\widehat{\mathbb{G}}$ is *unimodular*, that is, the left and right Haar weights of $\widehat{\mathbb{G}}$ coincide.
- Suppose $\mathbb{G}$ is of Kac type and $\kappa : C_0(\mathbb{G}) \to C_0(\mathbb{G})$ is the antipode. If $\tau : \mathcal{K}(\mathfrak{K}) \to \mathcal{K}(\bar{\mathfrak{K}})$ is the canonical anti-isomorphism, then $\overline{V} := (\tau \otimes \kappa)(V) \in M(\mathcal{K}(\bar{\mathfrak{K}}) \otimes C_0(\mathbb{G}))$ is a unitary corepresentation called the *contragredient* of V.

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- We can identify $\mathfrak{H} \otimes \bar{\mathfrak{K}}$ with the space of Hilbert-Schmidt operator space $HS(\mathfrak{K}, \mathfrak{H})$ from $\mathfrak{K}$ to $\mathfrak{H}$ through a linear isomorphism $\Phi : \mathfrak{H} \otimes \bar{\mathfrak{K}} \to HS(\mathfrak{K}, \mathfrak{H})$ defined, for all $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{K}$, by $\Phi(\xi \otimes \eta)(\zeta) = \langle \zeta, \eta \rangle \xi$ $(\forall \zeta \in \mathfrak{K})$, where the inner product on the right is taken in $\mathfrak{K}$. This yields an isomorphism $\mathrm{Ad}\,\Phi : \mathcal{L}(\mathfrak{H} \otimes \bar{\mathfrak{K}}) \to \mathcal{L}(HS(\mathfrak{K} \otimes \mathfrak{H}))$ defined by $\mathrm{Ad}\,\Phi(T) = \Phi T \Phi^*$ for all $T \in \mathcal{L}(\mathfrak{H} \otimes \bar{\mathfrak{K}})$.

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Then we have

Lemma

Suppose that $\mathbb{G}$ is of Kac type. $T \in \mathfrak{H} \otimes \bar{\mathfrak{K}}$ is $U \textcircled{\scriptsize T} \overline{V}$-invariant if and only if $U(T \otimes 1)V^* = T \otimes 1$ (as operators from $\mathfrak{K} \otimes L^2(\mathbb{G})$ to $\mathfrak{H} \otimes L^2(\mathbb{G})$).

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$\mathbb{G}$ is a locally compact quantum group of Kac type.

Proposition

$\pi_{\mathbf{1}_{\mathbb{G}}} \subset \pi_{U \textcircled{\top} \overline{V}}$ if and only if there exists a finite dimensional unitary corepresentation. W s.t. $\pi_W \subset \pi_U$ and $\pi_W \subset \pi_V$.

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Following the lemmas and propositions above, we obtain

Theorem

Property T of $\mathbb{G}$ is also equivalent to the following statements.

(4) *All finite dimensional irreducible representations of $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ are isolated points in $\widehat{C_0^{\mathrm{u}}(\widehat{\mathbb{G}})}$.*

(5) *$C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \cong B \oplus M_n(\mathbb{C})$ for a C^*-algebra B and $n \in \mathbb{N}$.*

Examples through bicrossed products

Next, I will give a non-trivial example of a discrete quantum group with property T.

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- G_1, G_2 : closed subgroups of G s.t. the canonical map from $G_1 \times G_2$ to G is a bijective homeomorphism

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• α, β : the canonical continuous actions of G_1 and G_2 on the right coset space $G_1 \backslash G$ and the left coset space G/G_2, respectively, that is,

$$\alpha(s)(G_1 x) = G_1 x s^{-1} \text{ and } \ \beta(t)(xG_2) = txG_2,$$

for all $s \in G_1$, $t \in G_2$, $G_1 x \in G/G_1$, $xG_2 \in G/G_2$.

Examples through bicrossed products

The following lemma easily follows from Proposition 1.1 in the paper “S. Baaj and G. Skandalis, Transformations pentagonales, *C. R. Acad. Sci. Paris Sr. I Math.* **327** (1998), 623-628".

Lemma

There exists a discrete quantum grou $\mathbb{G}$ *(called the bicrossed product of* G_1 *and* G_2*) such that* $C_0(\mathbb{G}) = C_0(G/G_2) \rtimes_{\beta,r} G_2$ *and* $C_0(\widehat{\mathbb{G}}) = C_0(G_1 \backslash G) \rtimes_{\alpha,r} G_1$.

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Then we have the main result in this section.

Theorem

If G_1 is discrete has property T and G_2 is finite, then $\mathbb{G}$ has property T.

Problem

It's well known that, if G is σ-compact, then property T of G is also equivalent to:

(T6) (Delorme-Guichardet Theorem) The first cohomology of G with coefficient in $(\pi, \mathfrak{H})$ vanishes, i.e., $H^1(G, \pi) = (0)$, for any strongly continuous unitary representation $(\pi, \mathfrak{H})$ of G.

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For any general locally compact quantum group $\mathbb{G}$,

- Delorme-Guichardet type theorem for $\mathbb{G}$?

A tentative answer

- X. Chen, A.T.M. Lau and C.K. Ng, *A property for locally convex *-algebras related to property T and character amenability*, preprint.

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In short,

1. we obtain a cohomological property for $C_c(G)$ (G is a locally compact group), which coincides with Kyed's D-G type theorem when G is a discrete group;

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1. we obtain a cohomological property for $C_c(G)$ (G is a locally compact group), which coincides with Kyed's D-G type theorem when G is a discrete group;
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3. an application to the theory of the fixed points.

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Thank you for your attention!

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