Property $T$ for locally compact quantum groups

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(joint works with Prof. Chi-Keung Ng)
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Background

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- P. Fima partially extended the notion of property $T$ to discrete quantum groups and he showed that discrete quantum groups with property $T$ are of Kac type.
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- P. Fima partially extended the notion of property $T$ to discrete quantum groups and he showed that discrete quantum groups with property $T$ are of Kac type.
- D. Kyed and M. Soltan partially generalize some equivalences of property $T$ to discrete quantum groups.
- M. Daws, P. Fima, A. Skalski and S. White gave the definition of property $T$ for general locally compact quantum groups, in their paper on the Haagerup property.
Notations and preliminaries on property $T$:

- If $G$ is a locally compact group and $\mu : G \to \mathcal{L}(\mathcal{H})$ is a continuous unitary representation, then a net $\{\xi_i\}_{i \in I}$ of unit vectors in $\mathcal{H}$ is called an *almost invariant unit vector (for $\mu$)* if

$$\sup_{t \in K} \|\mu_t(\xi_i) - \xi_i\| \to 0,$$

for any compact subset $K \subseteq G$. 

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- We denote by $C^*(G)$ (or $C_r^*(G)$) the full (or reduced) group $C^*$-algebra of $G$ and by $\pi_\mu$ the *-representation of $C^*(G)$ induced by $\mu$. 

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- $1_G$ is the trivial one-dimensional representation of $G$.

- $\hat{G}$ is the topological space of unitarily equiv. classes of irreducible unitary representations of $G$. 
• $G$ is said to have *property T* if any continuous unitary representation of $G$ that has an almost invariant unit vector actually has an invariant unit vector.
Preliminary

- $G$ is said to have **property $T$** if any continuous unitary representation of $G$ that has an almost invariant unit vector actually has an invariant unit vector.
- Property $T$ of $G$ is equivalent to:
  - (T1) $C^*(G) \cong \ker \pi_1 G \oplus \mathbb{C}$ canonically.
  - (T2) There exists a minimal projection $p \in M(C^*(G))$ such that $\pi_1 G(p) = 1$.
  - (T3) $1_G$ is an isolated point in $\hat{G}$.
  - (T4) All fin. dim. representations in $\hat{G}$ are isolated points.
  - (T5) There exists a fin. dim. representation in $\hat{G}$ which is an isolated point.
Preliminary

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  (T5) \( \exists \) a fin. dim. representation in $\hat{G}$ which is an isolated point.

• The equivalences of property $T$ with (T1)-(T5) for discrete quantum groups are some of the main results in the paper by Kyed and Soltan.
Property $T$ for locally compact quantum groups

Notations and preliminaries on locally compact quantum groups (all tensor product are spatial):

- $(C_0(G), \Delta, \varphi, \psi)$ is the reduced $C^*$-algebraic presentation of a locally compact quantum group $G$:
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  - \( \varphi \) is a faithful weight which is left invariant:
    “\((\text{id} \otimes \varphi)\Delta(a) = \varphi(a)1\)” (imprecise) and has faithful \( W^* \)-lift.

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- If $G$ is a locally compact group $G$, then $\Delta(f)(s, t) = f(st)$; $\varphi(f) = \int f(t) d\lambda(t)$ (for $f \in L^1(G) \cap C_0(G)$); $\psi$ is induced by the right Haar measure.

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Property $T$ for locally compact quantum groups

- **Unitary corepresentation**: is a unitary $U \in M(\mathcal{K}(\mathfrak{H}) \otimes C_0(\mathbb{G}))$ satisfying $(\text{id} \otimes \Delta)(U) = U_{12} U_{13}$, where $\mathcal{K}(\mathfrak{H})$ is the algebra of all compact operators on $\mathfrak{H}$. 
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- $1_{\mathbb{G}} \in M(C_0(\mathbb{G}))$: is the identity corepresentation.

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- $(C^u_0(\widehat{\mathbb{G}}), \Delta^u)$: the universal $C^*$-bialgebra associated with the dual group $\widehat{\mathbb{G}}$ of $\mathbb{G}$.
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- $1_\mathbb{G} \in M(C_0(\mathbb{G}))$: is the identity corepresentation.
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- $(C_0^u(\hat{\mathbb{G}}), \hat{\Delta}^u)$: the universal $C^*$-bialgebra associated with the dual group $\hat{\mathbb{G}}$ of $\mathbb{G}$.
- $V_G^u \in M(C_0^u(\hat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$: is a unitary that implements the bijective correspondence $U \leftrightarrow \pi_U$ between the set of all unitary corepresentations of $C_0(\mathbb{G})$ and the set of all non-degenerate $*$-representations of $C_0^u(\hat{\mathbb{G}})$ through the relation $U = (\pi_U \otimes \text{id})(V_G^u)$. 
Some notations and supplementary concerning a $C^*$-algebra $A$: 

- $\hat{A} \subseteq \text{Rep}(A)$: the set of unitarily equivalent classes of all irreducible representations of $A$ (equipped with Fell topology). 

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- **\( \text{Rep}(A) \)**: is the set of unitary equivalence classes of non-degenerate \(*\)-representations of a \( C^* \)-algebra \( A \).
- If \( (\mu, \mathcal{H}), (\nu, \mathcal{K}) \in \text{Rep}(A) \), we say
  - \( \mu \) is *contained* in \( \nu \), denoted by \( \mu \subset \nu \), if there exists an isometry \( V : \mathcal{H} \to \mathcal{K} \) s.t. \( \mu(a) = V^* \nu(a) V \), \( \forall a \in A \);
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- **$\hat{A} \subseteq \text{Rep}(A)$**: the set of unitarily equiv. classes of all irreducible representations of $A$ (equipped with Fell topology).
Lemma

Let $A$ be a $C^*$-algebra and $(\mu, \mathcal{H}) \in \hat{A}$.

(a) The following statements are equivalent.

1. $\mu$ is an isolated point in $\hat{A}$.
2. $\forall (\pi, \mathcal{K}) \in \text{Rep}(A) \text{ with } \mu \prec \pi$, one has $\mu \subset \pi$.
3. $A = \ker \mu \oplus \bigcap_{\nu \in \hat{A} \setminus \{\mu\}} \ker \nu$.

(b) If $\dim \mathcal{H} < \infty$, then $\{\mu\}$ is a closed subset of $\hat{A}$. 

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- For any unitary corepresentation $U \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(G))$,
  (a) $\xi \in \mathcal{H}$ is a $U$-invariant vector if $U(\xi \otimes \eta) = \xi \otimes \eta$, $\forall \eta \in L^2(G)$. 
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  (b) A net $\{\xi_i\}_{i \in \mathcal{I}}$ of unit vectors in $\mathcal{H}$ is called an almost $U$-invariant unit vector if $\|U(\xi_i \otimes \eta) - \xi_i \otimes \eta\| \to 0$, $\forall \eta \in L^2(G)$.
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- $U$ has a non-zero invariant vector $\iff \pi_{1\mathbb{G}} \subset \pi_U$;
Property $T$ for locally compact quantum groups

- For any unitary corepresentation $U \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(G))$,
  
  (a) $\xi \in \mathcal{H}$ is a \textit{U-invariant vector} if $U(\xi \otimes \eta) = \xi \otimes \eta$, $\forall \eta \in L^2(G)$.
  
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- $U$ has a non-zero invariant vector $\iff \pi_{1_G} \subset \pi_U$;
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**Definition**

$\mathbb{G}$ is said to have property $T$ if every unitary corepresentation of $C_0(\mathbb{G})$ having an almost invariant unit vector has a non-zero invariant vector.
The following extends some results in the paper by Kyed and Soltan:

**Proposition**

Property $T$ of a L.C.Q.G. $G$ is equivalent to:

1. $C_u^0(\hat{G}) \cong \ker \pi_1 G \oplus \mathbb{C}$.
2. There exists a projection $p_G \in M(C^u_0(\hat{G}))$ with $p_G C^u_0(\hat{G}) p_G = \mathbb{C} p_G$ and $\pi_1 G (p_G) = 1$.
3. $\pi_1 G$ is an isolated point in $C^u_0(\hat{G})$. 
Recall:

Let $G$ be a locally compact group.  

- Property $T$ of $G$ is equivalent to:

  (T1) $C^*(G) \cong \ker \pi_{1G} \oplus \mathbb{C}$ canonically.

  (T2) $\exists$ minimal projection $p \in M(C^*(G))$ such that $\pi_{1G}(p) = 1$.

  (T3) $1_{G}$ is an isolated point in $\hat{G}$.

  (T4) All fin. dim. representations in $\hat{G}$ are isolated points.

  (T5) $\exists$ a fin. dim. representation in $\hat{G}$ which is an isolated point.

For the corresponding results of (T4) and (T5), we need the notion of “tensor products” and “contragredients” of unitary corepresentations.
• If $V$ is another unitary corepresentation of $C_0(G)$ on a Hilbert space $\mathcal{K}$, then $U \boxtimes V := U_{13} V_{23} \in M(\mathcal{K}(\mathcal{H} \otimes \mathcal{K}) \otimes C_0(G))$ is a unitary corepresentation called the tensor product of $U$ and $V$. 

Recall that $G$ is of Kac type if its antipode is bounded, or equivalently, $\hat{G}$ is unimodular, that is, the left and right Haar weights of $\hat{G}$ coincide. 

Suppose $G$ is of Kac type and $\kappa : C_0(G) \to C_0(G)$ is the antipode. If $\tau : K(\overline{\mathcal{K}}) \to K(\overline{\mathcal{K}})$ is the canonical anti-isomorphism, then $V := (\tau \otimes \kappa)(V) \in M(K(\overline{\mathcal{K}}) \otimes C_0(G))$ is a unitary corepresentation called the contragredient of $V$. 

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- If $V$ is another unitary corepresentation of $C_0(G)$ on a Hilbert space $\mathcal{H}$, then $U \boxplus V := U_{13}V_{23} \in M(\mathcal{K}(\mathcal{H} \otimes \mathcal{K}) \otimes C_0(G))$ is a unitary corepresentation called the tensor product of $U$ and $V$.

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- We can identify \(\mathcal{H} \otimes \bar{\mathcal{K}}\) with the space of Hilbert-Schmidt operator space \(HS(\mathcal{K}, \mathcal{H})\) from \(\mathcal{K}\) to \(\mathcal{H}\) through a linear isomorphism \(\Phi: \mathcal{H} \otimes \bar{\mathcal{K}} \rightarrow HS(\mathcal{K}, \mathcal{H})\) defined, for all \(\xi \in \mathcal{H}\) and \(\eta \in \mathcal{K}\), by
  \[
  \Phi(\xi \otimes \eta)(\zeta) = \langle \zeta, \eta \rangle \xi \quad (\forall \zeta \in \mathcal{K}),
  \]
  where the inner product on the right is taken in \(\mathcal{K}\). This yields an isomorphism
  \(Ad \Phi: \mathcal{L}(\mathcal{H} \otimes \bar{\mathcal{K}}) \rightarrow \mathcal{L}(HS(\mathcal{K} \otimes \mathcal{H}))\)
  defined by
  \[
  Ad \Phi(T) = \Phi T \Phi^* \quad \text{for all} \quad T \in \mathcal{L}(\mathcal{H} \otimes \bar{\mathcal{K}}).
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We can identify $\mathcal{H} \otimes \bar{\mathcal{K}}$ with the space of Hilbert-Schmidt operator space $HS(\mathcal{K}, \mathcal{H})$ from $\mathcal{K}$ to $\mathcal{H}$ through a linear isomorphism $\Phi : \mathcal{H} \otimes \bar{\mathcal{K}} \rightarrow HS(\mathcal{K}, \mathcal{H})$ defined, for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, by $\Phi(\xi \otimes \eta)(\zeta) = \langle \zeta, \eta \rangle \xi$ ($\forall \zeta \in \mathcal{K}$), where the inner product on the right is taken in $\mathcal{K}$. This yields an isomorphism $\text{Ad } \Phi : \mathcal{L}(\mathcal{H} \otimes \bar{\mathcal{K}}) \rightarrow \mathcal{L}(HS(\mathcal{K} \otimes \mathcal{H}))$ defined by $\text{Ad } \Phi(T) = \Phi T \Phi^*$ for all $T \in \mathcal{L}(\mathcal{H} \otimes \bar{\mathcal{K}})$. Then we have

**Lemma**

*Suppose that $G$ is of Kac type. $T \in \mathcal{H} \otimes \bar{\mathcal{K}}$ is $U \bigoplus \bar{V}$-invariant if and only if $U(T \otimes 1)V^* = T \otimes 1$ (as operators from $\mathcal{K} \otimes L^2(G)$ to $\mathcal{H} \otimes L^2(G)$).*
Property $T$ for locally compact quantum groups

$G$ is a locally compact quantum group of Kac type.

**Proposition**

$\pi_{1_G} \subset \pi_{U \overline{\otimes} V}$ if and only if there exists a finite dimensional unitary corepresentation $W$ s.t. $\pi_W \subset \pi_U$ and $\pi_W \subset \pi_V$.
Proposition

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Following the lemmas and propositions above, we obtain

Theorem

Property $T$ of $G$ is also equivalent to the following statements.

(4) All finite dimensional irreducible representations of $C^u_0(\hat{G})$ are isolated points in $C^u_0(\hat{G})$.

(5) $C^u_0(\hat{G}) \cong B \oplus M_n(\mathbb{C})$ for a $C^*$-algebra $B$ and $n \in \mathbb{N}$.
Next, I will give a non-trivial example of a discrete quantum group with property $T$. 
Examples through bicrossed products

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- $G$: a locally compact group.
- $G_1$, $G_2$: closed subgroups of $G$ s.t. the canonical map from $G_1 \times G_2$ to $G$ is a bijective homeomorphism

\[ \gamma : G_1 \times G_2 \to G, \ (g_1, g_2) \mapsto g_1 g_2 \]

onto to an open dense subset $\Omega$ of $G$. 
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\[ \gamma : G_1 \times G_2 \rightarrow G, \ (g_1, g_2) \mapsto g_1 g_2 \]

onto to an open dense subset $\Omega$ of $G$.
- $\alpha, \beta$: the canonical continuous actions of $G_1$ and $G_2$ on the right coset space $G_1 \backslash G$ and the left coset space $G/G_2$, respectively, that is,

\[ \alpha(s)(G_1 x) = G_1 xs^{-1} \quad \text{and} \quad \beta(t)(xG_2) = txG_2, \]

for all $s \in G_1, t \in G_2, G_1 x \in G/G_1, xG_2 \in G/G_2$. 
Examples through bicrossed products


**Lemma**

There exists a discrete quantum group $G$ (called the bicrossed product of $G_1$ and $G_2$) such that $C_0(G) = C_0(G/G_2) \rtimes_{\beta, r} G_2$ and $C_0(\hat{G}) = C_0(G_1 \setminus G) \rtimes_{\alpha, r} G_1$. 

**Lemma**

There exists a discrete quantum group $\mathbb{G}$ (called the bicrossed product of $G_1$ and $G_2$) such that $C_0(\mathbb{G}) = C_0(G/G_2) \rtimes_{\beta,r} G_2$ and $C_0(\hat{\mathbb{G}}) = C_0(G_1 \setminus G) \rtimes_{\alpha,r} G_1$.

Then we have the main result in this section.

**Theorem**

If $G_1$ is discrete has property $T$ and $G_2$ is finite, then $\mathbb{G}$ has property $T$. 
It’s well known that, if $G$ is $\sigma$-compact, then property $T$ of $G$ is also equivalent to:

(T6) (Delorme-Guichardet Theorem) The first cohomology of $G$ with coefficient in $(\pi, \mathcal{H})$ vanishes, i.e., $H^1(G, \pi) = (0)$, for any strongly continuous unitary representation $(\pi, \mathcal{H})$ of $G$. 
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For any general locally compact quantum group $G$,
- Delorme-Guichardet type theorem for $G$?
A tentative answer

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In short,

1. we obtain a cohomological property for $C_c(G)$ ($G$ is a locally compact group), which coincides with Kyed’s D-G type theorem when $G$ is a discrete group;

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3. an application to the theory of the fixed points.
References


Thank you for your attention!

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