Time for graph bootstrap percolation

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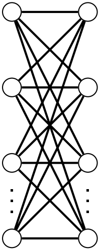
Prairie Discrete Math Workshop
8 August 2015

Joint work with S. Koch (Cambridge) and M. Przykucki (Oxford)

K_3-free graphs

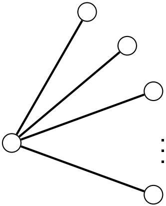
Among all K_3-free graphs on n vertices with the property that adding any missing edge creates a triangle, those with the

Maximum number of edges:



$K_{n/2,n/2}$ - Mantel's theorem (1907)

Minimum number of edges:

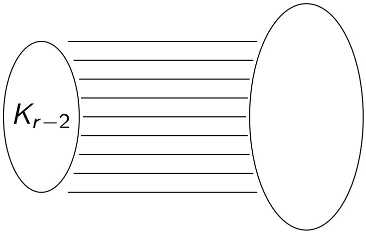


$K_{1,n-1}$

K_r-free graphs

Replace K_3 with K_r (complete graph on r vertices).

- Maximum number of edges: Turán's theorem (1941) $K_{n/(r-1),n/(r-1),\ldots,n/(r-1)}$.
- Minimum number of edges: Erdős, Hajnal, Moon (1964) $K_{1,1,\ldots,1,n-r+2}$.



$n-r+2$ indep. vxs

Graph Bootstrap

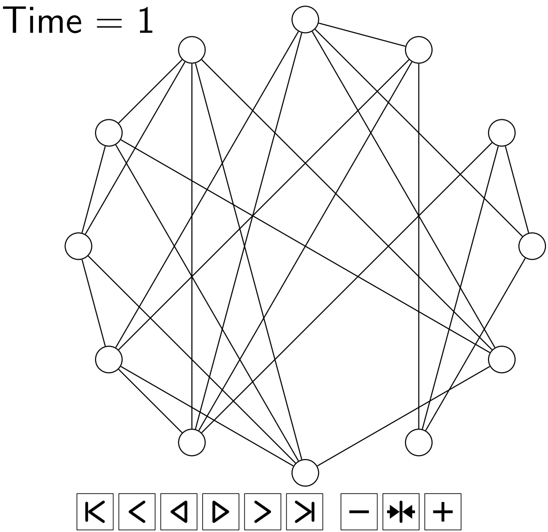
Given graphs G, F, define the *F-bootstrap process on G*:

- $G_0 = G$, and
- for $t \geq 0$,

$$G_{t+1} = G_t \cup \{e \mid \text{ adding } e \text{ to } G_t \text{ creates a new copy of } F\}.$$

G *percolates* if for some t_0, G_{t_0} is complete.
Introduced by Bollobás (1968) under the name *weak-saturation*.

K_4-bootstrap process



Extremal results

Theorem (Frankl 1982, Kalai 1984, Alon 1985)

All $r \geq 3$: If G is a graph on n vertices that percolates in K_r-bootstrap, then

$$e(G) \geq \binom{r-2}{2} + (r-2)(n-r+2).$$

Same as lower bound for K_r-saturated graphs!

Random graphs

$G_{n,p}$ = Erdős-Rényi random graph
If $G = G_{n,p}$, what is

$$\mathbb{P}_p(G \text{ percs in } F\text{-bootstrap}) = ?$$

Definition

Critical probability for F-bootstrap percolation:

$$p_c(n, F) = \inf\{p \mid \mathbb{P}_p(G_{n,p} \text{ percs in } F\text{-bootstrap}) \geq 1/2\}.$$

K_3-percolation

Example: When $F = K_3$, a graph G percolates in the K_3-process iff G is connected.

Thus, by result of Erdős, Rényi (1959)

$$p_c(n, K_3) = \frac{\log n}{n}(1 + o(1)).$$

Critical probability thresholds

Define

$$\lambda(r) = \frac{\binom{r}{2} - 2}{r - 2} = \frac{r+1}{2} - \frac{1}{r-2}.$$

Theorem (Balogh, Bollobás, Morris 2012)

For every $r \geq 4$, there is a constant $c(r) > 0$ so that

$$\frac{n^{-1/\lambda(r)}}{c(r)\log n} \leq p_c(n, K_r) \leq n^{-1/\lambda(r)} \log n.$$

And, further, for $r = 4$,

$$\frac{1}{4}\sqrt{\frac{1}{n \log n}} \leq p_c(n, K_4) \leq 10\sqrt{\frac{1}{n \log n}}.$$

Time

When G percolates in the F-bootstrap process, how long does it usually take?
Example: $F = K_3$

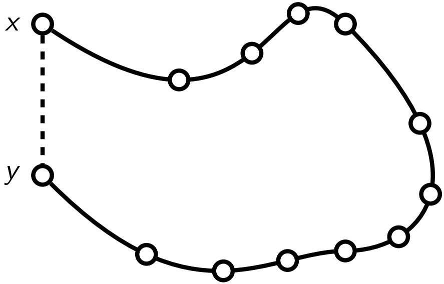
If G is connected, time to K_3-percolation:

$$\lceil \log_2(\text{diam}(G)) \rceil.$$

Fastest: $K_{1,n}$, time to K_3-percolation $= 1$.
Slowest: P_n, time to K_3-percolation $= \lceil \log_2 n \rceil$.

Random graphs

For which ranges of $p(n)$ might one expect $G_{n,p(n)}$ to K_3-percolate by time t?



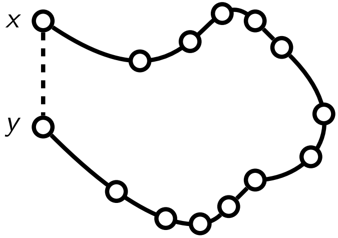
Path of length $\leq 2^t$

If $p = p(n) \leq n^{-1+1/2^t} / \log n$, then for any vertices x and y:

$$\begin{aligned}
&\mathbb{P}(x,\ y \text{ joined by path of length } \leq 2^t) \\
&\leq \sum_{k=1}^{2^t} \mathbb{P}(x,\ y \text{ joined by path of length } k) \\
&\leq \sum_{k=1}^{2^t} \binom{n-2}{k-1}(k-1)!p^k \leq \sum_{k=1}^{2^t} n^{k-1}p^k \\
&\leq \sum_{k=1}^{2^t} n^{k-1}\left(\frac{n^{-k+k/2^t}}{(\log n)^k}\right) \\
&= \sum_{k=1}^{2^t} \frac{n^{k/2^t-1}}{(\log n)^k} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

When $p(n) > n^{-1+1/2^t} \log n$, the average number of paths of length 2^t between x and y is huge.

Counting pairs of overlapping paths and showing that they are less likely to occur than pairs of disjoint paths and the second moment method show that with high probability, there is at least one such $x - y$ path.



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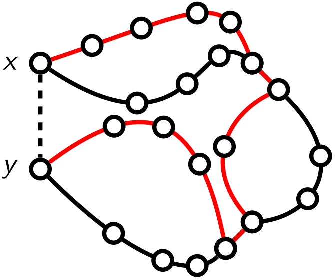


A much more precise result on the diameter of a random graph:

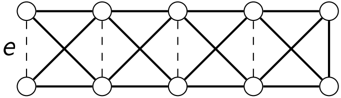
Theorem (Bollobás, 1981)

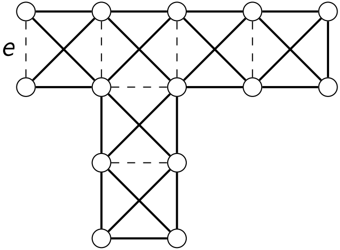
For $d \leq \frac{\log n}{4 \log \log n}$, if

$$(3 \log n)^{1/d} n^{-1+\frac{1}{d}} < p(n) < (\log n)^{1/(d-1)} n^{-1+\frac{1}{d-1}},$$

then, whp $G_{n,p(n)}$ has diameter d.

For $r \geq 4$, how long does K_r-percolation take?
Lots of ways to K_r-percolate in a fixed time:
Example: e added in time 4 in K_4-bootstrap

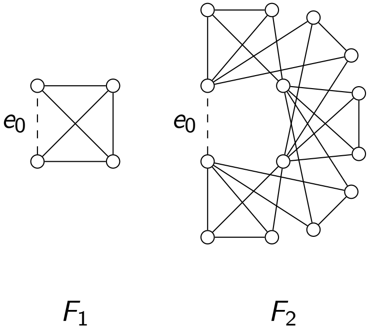


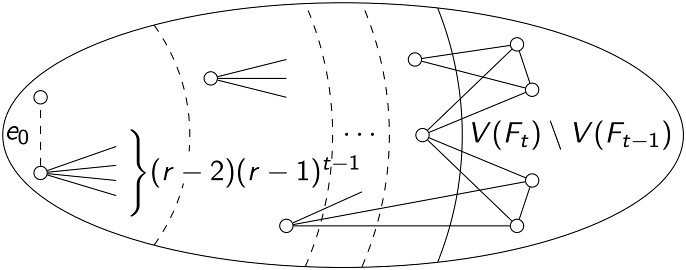


What is the most likely way for e to be added by time t?

Minimal infecting graph

Want to find the sparsest minimal graph that "witnesses" a particular edge being added at time t.





F_t

Picture due to M. Przykucki

Threshold for percolation by time t

Recall that $\lambda = \lambda(r) = \frac{\binom{r}{2}-2}{r-2}$.

Theorem (G, Koch, Przykucki, 2015)

For every $t \geq 1$, $r \geq 4$ there exist constants $c_t = c_t(r) > 0$ and $C = C(r) > 0$ so that if $t \leq C \log\log n$ then

- *if $p(n) \leq n^{-\frac{1}{\lambda(r)(1+c_t)}} \frac{1}{\log n}$, then*

$$\mathbb{P}(G_{n,p(n)} \ K_r\text{-percs by time } t) = o(1),$$

- *if $p(n) \geq n^{-\frac{1}{\lambda(r)(1+c_t)}} \log n$, then*

$$\mathbb{P}(G_{n,p(n)} \ K_r\text{-percs by time } t) = 1 - o(1).$$

Thank you!