Characterizing the degree sequences of hypergraphs

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Let $H(V, E)$ be a hypergraph. The **degree** of a vertex $v \in V$ is the number of edges incident with v.

The **degree sequence** of a hypergraph is the list of vertex degrees (usually) arranged in nonincreasing order.

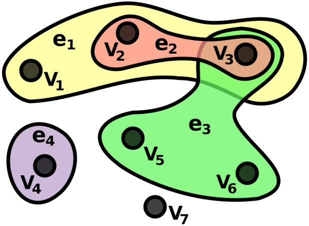

Figure 1 : A hypergraph $H(V, E)$ with $V = \{v_1, \ldots, v_7\}$ and $E = \{e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_5, v_6\}, e_4 = \{v_4\}\}$

The degree sequence is

$$[d_i]_{i=1}^{7} = d_1 \geq \cdots \geq d_7 = 3, 2, 1, 1, 1, 1, 0$$

Characterizing the degree sequences

A hypergraph is said to be **simple** if none of its edges is a subset of another.

Problem

Find Erdős–Gallai type necessary and sufficient conditions for a sequence of non-negative integers to be the degree sequence of a simple hypergraph (or a special class of hypergraphs).

Theorem 1 (Erdős–Gallai, 1960)

A sequence $[d_i]_{i=1}^n$ of non-negative integers arranged in nonincreasing order is the degree sequence of a simple graph on n vertices if and only if

1. $\sum_{i=1}^n d_i$ *is even, and*
2. *for every* $1 \leq j \leq n$,

$$\sum_{i=1}^{j} d_i \leq j(j-1) + \sum_{i=j+1}^{n} \min\{d_i, j\}.$$

Known results

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A **linear hypergraph** is one in which any two edges have at most one common vertex.

Theorem 2 (Bhave, Bam, Deshpande, 2009)

A sequence $[d_i]_{i=1}^n$ of non-negative integers arranged in nonincreasing order is the degree sequence of a linear hypergraph if and only if it satisfies the Erdős–Gallai conditions.

Known results

Let $k \geq 2$ be a positive integer. A hypergraph is said to be k-**uniform** if all its edges are incident with k vertices. A k-**hypergraph** is a simple k-uniform hypergraph.

A sequence is k-**graphic** if it is the degree sequence of a k-hypergraph.

Proposition 3 (Folklore)

If $[d_i]_{i=1}^n$ is k-graphic then

1. $\sum_{i=1}^n d_i \equiv 0 \pmod{k}$, *and*
2. *for every* $1 \leq j \leq n$,

$$\sum_{i=1}^{j} d_i \leq k\binom{j}{k} + (k-1)\sum_{i=j+1}^{n} d_i.$$

Billington (1988) and Choudum (1991) gave improved necessary conditions for 3-hypergraphs.

Known results

A **partial Steiner triple system (PSTS)** is a linear 3-hypergraph.

Theorem 4 (Keranen, Kocay, Kreher, Li, 2008)

If $d_1 \geq \cdots \geq d_n$ *is the degree sequence of a PSTS then*

1. $\sum_{i=1}^{n} d_i \equiv 0 \pmod 3$,
2. *for every* $1 \leq j \leq n/2$,

$$\sum_{i=1}^{j} d_i \leq \frac{3}{2}\binom{j}{2} + \frac{1}{2}\sum_{i=j+1}^{n} \min\{d_i, j\},$$

3. *for every* $n/2 < j \leq n$,

$$\sum_{i=1}^{j} d_i \leq \binom{j}{2} + \frac{n-j}{2}\left\lfloor \frac{j}{2} \right\rfloor + \frac{1}{2}\sum_{i=j+1}^{n} \min\{d_i, j\}.$$

Generalizing Theorems 2 and 4

A **partial** (n, k, λ)-**system** is a k-uniform hypergraphs with n vertices in which any pair of vertices occurs in at most λ common edges.

Partial (n, k, λ)-systems generalize graphs, linear hypergraphs and partial Steiner triple systems.

Theorem 5 (Khan, 2014)

If a sequence $[d_i]_{i=1}^n$ of non-negative integers arranged in nonincreasing order is the degree sequence of a partial (n, k, λ)-system then

1. $\sum_{i=1}^n d_i \equiv 0 \pmod{k}$, *and*
2. *for every* $1 \leq j \leq n$

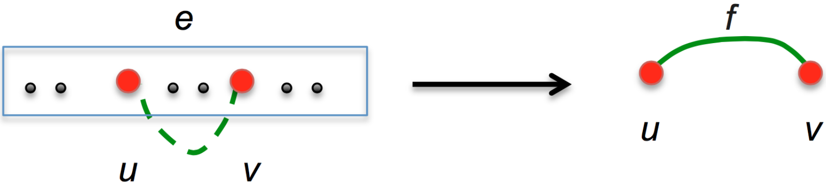
$$\sum_{i=1}^{j} d_i \leq \frac{\lambda j(j-1)}{k-1} + \frac{1}{k-1} \sum_{i=j+1}^{n} \min\{\lambda j, d_i\}. \tag{1}$$

Proof

An r-**multigraph** is a graph in which each pair of vertices is joined by at most r edges.

Let $H(V,E)$ be a partial (n,k,λ)-system with degree sequence $[d_i]_{i=1}^n$ arranged in noninccreasing order. Form an r-multigraph $G(V,F)$ from H as follows:

For every edge $e = \{\dots, u, \dots, v, \dots\} \in E$ insert an edges f joining u and v in G.



Then G is a λ-multigraph with degree sequence $[D_i]_{i=1}^n = [(k-1)d_i]_{i=1}^n$.

Proof

Chungphaisan (1974) showed that $[D_i]_{i=1}^n$ is the degree sequence of an r-multigraph if and only if $\sum_{i=1}^n D_i$ is even and for every $1 \leq j \leq n$,

$$\sum_{i=1}^{n} D_i \leq rj(j-1) + \sum_{i=j+1}^{n} \min\{rj, d_i\}.$$

Substituting $r = \lambda$ and $D_i = (k-1)d_i$ gives

$$\sum_{i=1}^{j} d_i \leq \frac{\lambda j(j-1)}{k-1} + \frac{1}{k-1} \sum_{i=j+1}^{n} \min\{\lambda j, d_i\}.$$

□

A **tournament** is a complete oriented graph and a k-**hypertournament** is a complete oriented k-hypergraph. Tournaments are 2-hypertournaments.

The **score** $s(v)$ of a vertex v in a k-hypertournament is the number of arcs containing v but not as the last element.

The **score sequence** is formed by listing the vertex scores (usually) in nondecreasing order.

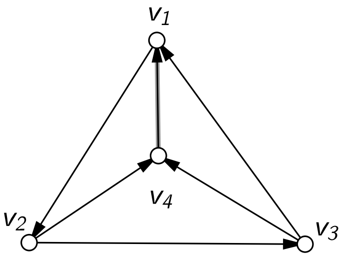


Figure 2 : A tournament with score sequence $[s_i]_{i=1}^{4} = s_1 \leq \cdots \leq s_4 = 1, 1, 2, 2$

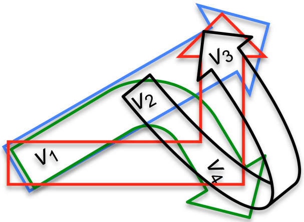


Figure 3 : A 3-hypertournament $H(V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1 = (v_1, v_2, v_3), e_2 = (v_1, v_2, v_4), e_3 = (v_1, v_4, v_3), e_4 = (v_2, v_4, v_3)\}$ and score sequence $0, 2, 3, 3$.

For more on hypertournaments see

- K. Kayibi, M. A. Khan, S. Pirzada, Uniform sampling of k-hypertournaments, *Linear Multilinear Algebra* **61** (2013), No. 1, 123–138.
- M. A. Khan, S. Pirzada, K. Kayibi, Scores, inequalities and regular hypertournaments, *Math. Inequal. Appl.* **15** (2012), No. 2, 343–351.

Theorem 6 (Landau, 1953)

A nondecreasing sequence $S = [s_i]_{i=1}^n$ of non-negative integers is the score sequence of some tournament if and only if for each $1 \leq j \leq n$,

$$\sum_{i=1}^{j} s_i \geq \binom{j}{2},$$

with equality when $j = n$.

Theorem 7 (Guofei, Yao, Zhang, 2000)

A nondecreasing sequence $S = [s_i]_{i=1}^n$ of non-negative integers is the score sequence of some k-hypertournament if and only if for each $1 \leq j \leq n$,

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k} \quad (= \text{Bound1}),$$

with equality when $j = n$.

A stronger characterization

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Theorem 8 (Pirzada, Khan, Guofei, Kayibi, 2015)

Given two non-negative integers n and k with $n \geq k > 1$, a sequence $[s_i]_{i=1}^{n}$ of non-negative integers in nondecreasing order is the score sequence of some k-hypertournament if and only if for every subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,

$$\sum_{i \in I} s_i \geq \frac{2k\,|I| - n}{2k} \binom{n-1}{k-1} + \frac{1}{2} \binom{n - |I|}{k} - \frac{1}{2} \sum_{i \in I} \binom{i-1}{k-1} \quad (= \text{Bound2})$$

with equality when $|I| = n$.

Proof appears (in terms of losing scores) in

S. Pirzada, M. A. Khan, Z. Guofei, K. Kayibi, On scores, losing scores and total scores in k-hypertournaments, *Electron. J. Graph Theory Appl.* **3** (2015), no. 1, 8–21.

A simpler characterization

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Theorem 9 (Khan, 2015)

A sequence $[s_i]_{i=1}^n$ of non-negative integers, arranged in non-decreasing order, is the score sequence of a hypertournament if and only if for each $1 \leq j \leq n$,

$$\sum_{i=1}^{j} s_i \geq \frac{2}{k}\binom{n-2}{k-2}\binom{j}{2}, \quad (= \text{Bound3})$$

with equality when $j = n$.

Note: Bound2 $\geq$ Bound1 $\geq$ Bound3

Proof of Theorem 9 will appear in

M. A. Khan, *k*-hypertournament matrices revisited, *preprint*.