Vertex-primitive digraphs having vertices with almost equal neighbourhoods

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If $\Gamma$ is symmetric then we call it a **graph**.
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Note that vertex-transitive digraphs are regular. ($|\Gamma(v)|$ does not depend on $v$.)
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Vertex-primitive digraphs

Γ is vertex-primitive if it is vertex-transitive and $\text{Aut}(\Gamma)$ preserves no nontrivial partition of $\Omega$.

1. Disconnected digraphs are not vertex-primitive.
2. Connected bipartite graphs are not vertex-primitive.
3. $K_n$ is vertex-primitive.
4. $C_n$ is vertex-primitive if and only if $n$ is prime.
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A little more notation

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Note that $\Gamma_i$ really is a graph, and it is also vertex-primitive.

For example, $\Gamma_0$ is the graph with two vertices adjacent if they have the same neighbourhood.
Easy exercise: vertices with the same neighbourhood

Lemma

If $\Gamma$ is vertex-primitive and $\Gamma_0 \neq \Omega^*$ then $\Gamma = \emptyset$ or $\Gamma = \Omega \times \Omega$. 
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Proof.

$\Gamma_0$ is an equivalence relation preserved by a primitive group. Since $\Gamma_0 \neq \Omega^*$, $\Gamma_0 = \Omega \times \Omega$. 
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**Proof.**

$\Gamma_0$ is an equivalence relation preserved by a primitive group. Since $\Gamma_0 \neq \Omega^*$, $\Gamma_0 = \Omega \times \Omega$.

If $\Gamma \neq \emptyset$ there exists $(\alpha, \beta) \in \Gamma$. As $\Gamma_0 = \Omega \times \Omega$, all vertices of $\Gamma$ have the same neighbourhood and thus $\beta \in \Gamma(\omega)$ for every $\omega \in \Omega$ but then vertex-transitivity implies that $\Gamma = \Omega \times \Omega$. 

☐
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Examples
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- $C_n$ when $n$ is prime.

A computer search suggested that, apart from $K_n$ and $\Omega^*$, all examples have prime order.
$\Delta_{11,2,3}$:
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Theorem (Spiga, Verret, 2015)

If $\Gamma_1 \neq \emptyset$ then $\Gamma$ is one of $K_n$, $\Omega^*$ or $\Delta_{p,x,d}$, for some prime $p$. 
A consequence of a more general theorem

Let \( n \) be the order of \( \Gamma \). Let \( \kappa \) be the smallest positive \( i \) such that \( \Gamma^i \neq \emptyset \).
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Let $n$ be the order of $\Gamma$. Let $\kappa$ be the smallest positive $i$ such that $\Gamma_i \neq \emptyset$.

Theorem (Spiga, Verret, 2015)

If $\emptyset \neq \Gamma \neq \Omega \times \Omega$, then either

1. $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$ and $(n - 1)(d - \kappa) = d(d - 1)$, or

2. there exists $i \in \{\kappa, \ldots, d - 1\}$ such that $\Gamma_i$ has valency at least 1 and at most $\kappa^2 + \kappa$.
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(In the second case, we use the fact that a cycle is vertex-primitive if and only if it has prime order.)
The case $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$

We can show $n \leq \kappa^2 + \kappa + 1$ (apart from the trivial case $k \in \{1, d\}$).
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In particular, for any specific value of $\kappa$, this is a “finite” problem, with a somewhat “effective” solution.
Sketch of proof of main theorem

Let \( \ell := \min\{i \geq \kappa : \Gamma_{i+1} = \Gamma_{i+2} = \cdots = \Gamma_{i+\kappa} = \emptyset\} \).
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$$|\mathcal{S}| = \sum_{b, b' \in \mathcal{B}, b \neq b'} |b \cap b'| = \sum_{b, b' \in \mathcal{B}, b \neq b'} (d - \kappa) = n(n - 1)(d - \kappa).$$
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$$|\mathcal{S}| = \sum_{b, b' \in \mathcal{B}} |b \cap b'| = \sum_{b, b' \in \mathcal{B}} (d - \kappa) = n(n-1)(d - \kappa).$$

Therefore $(n-1)(d - \kappa) = d(d-1)$. 
Case II, $\ell \geq \kappa + 1$

Let $\alpha \in \Omega$ and let $S(\alpha) := \{(\beta, \gamma) \in \Omega \times \Omega : \beta \in \Gamma(\alpha) \cap \Gamma(\gamma)\}$. 

A slightly more complicated argument yields:

$$|S(\alpha)| \geq d_2^2 - \ell \left(\ell - 1\right) + \ell - 1 \sum_{i=1}^{\ell - 1} d_i \left(\ell - i\right),$$

where $d_i$ is the valency of $\Gamma_i$. From there, we find that, for some $i$, $d_i \leq \kappa_2^2 + \kappa_2$. 
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(For example, if $f(1, 2, 3, 4) = (2, 2, 3, 2)$ then $f$ has kernel type $(1, 3)$.)
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(For example, if $f(1, 2, 3, 4) = (2, 2, 3, 2)$ then $f$ has kernel type $(1, 3)$.)

We say that $G$ synchronises $f$ if the semigroup $\langle G, f \rangle$ contains a constant map, while $G$ is said to be synchronising if $G$ synchronises every non-invertible map on $\Omega$. 
Synchronising groups II

(Synchronising $\implies$ primitive) but the converse is not true.
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Theorem (Araújo, Cameron, 2014)

If $G$ is primitive and $f$ has kernel type $(2, 2, 1, \ldots, 1)$, then $G$ synchronises $f$.

They asked about the case $(3, 2, 1, \ldots, 1)$.

Theorem (Spiga, Verret, 2015)

If $G$ is primitive and $f$ has kernel type $(p, 2, 1, \ldots, 1)$ with $p \geq 2$, then $G$ synchronises $f$.

This was later proved independently by Araújo, Bentz, Cameron, Royle and Schaefer.
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The valency 6 case might also be doable, but the “little” work does not seem trivial.