

# Skew-shapes with interval support in the dominance lattice

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Banff, August 14-16, 2015

## Introduction

I. Classification of a direct sum of a class ribbon of Schur functions with interval support

II. Bad configurations. Full configurations. Classification of multiplicity free skew Schur functions with interval support

# Skew Schur functions

- ▶ Schur functions are considered to be the most important basis for the ring of symmetric functions.

# Skew Schur functions

- ▶ Schur functions are considered to be the most important basis for the ring of symmetric functions.
- ▶ Let  $x = (x_1, x_2, \dots)$ . Given partitions  $\mu \subseteq \lambda$ ,  $A := \lambda/\mu$ .  
The skew-Schur function  $s_A$  is the generating function for SSYT  $T$  of shape  $A$

$$s_A(x) = \sum_T x^T,$$

where the sum is over all SSYT  $T$  of shape  $A$ . Thus it is a symmetric function.



$$s_A = \sum_{\nu} c_{A, \nu}^{\nu} s_{\nu},$$

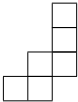
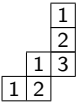
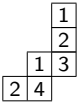
where  $c_{A, \nu}^{\nu} := c_{\mu, \lambda}^{\nu} \geq 0$  is the number of SSYT of shape  $A$  and content  $\nu$ , satisfying the Littlewood-Richardson rule.

# Skew Schur functions

►  $c_A^\nu = c_{A'}^{\nu'}$

$$s_A = \sum_{c(A) \preceq \nu' \preceq r(A)'} c_{\mu, \lambda}^\nu s_\nu = s_{r(A)} + \cdots + c_A^\nu s_\nu + \cdots + s_{c(A)'} = s_{A^\pi}$$

$$s_{A'} = \sum_{c(A) \preceq \nu' \preceq r(A)'} c_{\mu, \lambda}^{\nu'} s_{\nu'} = s_{c(A)} + \cdots + c_A^{\nu'} s_{\nu'} + \cdots + s_{r(A)'}$$

$A =$ 

 $c(A) =$ 

 $, \quad c(A) = 321, \quad r(A)' =$ 


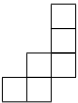
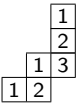
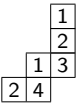
$$[321, 42] = \{321, \mathbf{33}, 411, 42\}.$$

# Skew Schur functions

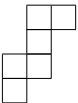
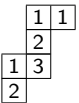

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$A =$ 

 $c(A) =$ 

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 $r(A)' =$ 

 $42$

$$[321, 42] = \{321, \mathbf{33}, 411, 42\}.$$

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 $$ 

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# Skew Schur function support

- ▶ The support of a skew shape  $A$ ,  $\text{supp}A$ , considered as a subposet of the *dominance lattice*, has a top element and a bottom element uniquely defined by the shape  $A$ ,

$$r(A), c(A)' \in \text{supp}A = \{\nu : c_A^\nu > 0\} \subseteq [r(A), c(A)']$$

$$c(A), r(A)' \in \text{supp}A' = \{\nu' : c_A^{\nu'} > 0\} \subseteq [c(A), r(A)']$$

$$\text{supp}A = \text{supp}A^\pi$$

$$c_A^{c(A)} = c_A^{r(A)'} = 1$$

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- ▶ The support of  $s_A$  is the support of  $A$ .



# Problems

Given the skew shape  $A$  and  $\nu' \in [c(A), r(A)']$

1. How does the shape of  $A$  govern the positivity of  $c_A^{\nu'}$ ?  
How does the shape of  $A$  govern the support of  $A$ ?

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2. Under what conditions do we have  $c_A^{\nu'} > 0$  whenever  $\nu' \in [c(A), r(A)']$ ?

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2. Under what conditions do we have  $c_A^{\nu'} > 0$  whenever  $\nu' \in [c(A), r(A)']$ ?

Which skew shapes have interval support?

A., *The admissible interval for the invariant factors of a product of matrices*,  
Linear and Multilinear Algebra (1999).

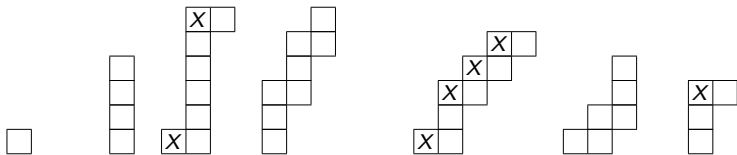
*If  $A$  is a skew shape with two or more components and  $A$  has interval support, then the components of  $A$  are ribbon shapes.*

3 Which are the ribbon shapes with interval support?

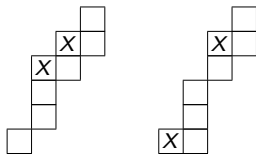
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► **Answer.** Ribbons whose column (row) lengths are at least two except possibly the top and bottom columns (rows).



*Direct sums of similar ribbons*



**Example.** (Direct sum) Ribbons such that all columns and row lengths differ by at most one.

# Dominance order on partitions

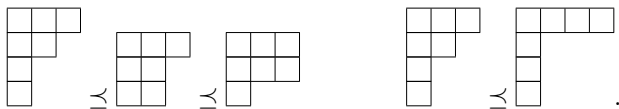
- ▶ The *dominance order*  $\preceq$  on partitions of  $N$ ,  $\lambda = (\lambda_1, \dots, \lambda_l)$ ,  $\mu = (\mu_1, \dots, \mu_s)$  is defined by setting  $\lambda \preceq \mu$  if

$$\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i,$$

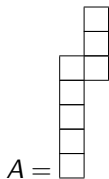
for  $i = 1, \dots, l$ , where we set  $\mu_i = 0$  if  $i > l$ .

The set of partitions of size  $N$  equipped with the dominance order is a lattice with maximum element  $(N)$  and minimum element  $(1^N)$ .

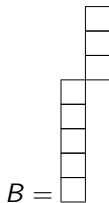
- ▶  $\lambda \preceq \mu$  if and only if the Young diagram of  $\mu$  is obtained by "*lifting*" at least one box in the Young diagram of  $\lambda$ .



# Skew Schur functions and support



$$A = s_{A'} = s_{53} + s_{62} + s_{71}$$



$$B = s_{B'} = s_{53} + s_{62} + s_{71} + s_8$$

$$r(A)' = 71$$


$$62$$

$$c(A) = 53$$



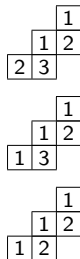


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
►  $A =$  

$$s_{A'} = s_{221} + s_{311} + s_{32}$$

$$\begin{array}{c} r(A)' = 32 \\ | \\ 311 \\ | \\ c(A) = 221 \end{array}$$



# Skew Schur functions and support

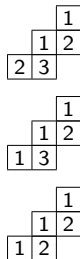
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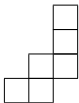
$$311$$

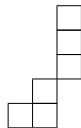
$$c(A) = 221$$



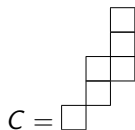
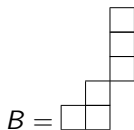
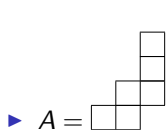
$$0 \leq 1 - 1, \quad 1 \leq 2 + 1 - 2$$

# Skew Schur functions and support

►  $A =$  

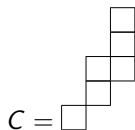
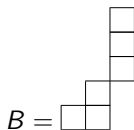
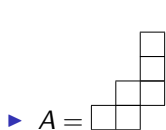
$B =$  

# Skew Schur functions and support



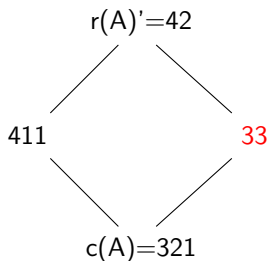
$$s_{A'} = s_{321} + s_{411} + 0s_{33} + s_{42}$$
$$s_{B'} = s_{321} + s_{411} + 0s_{33} + s_{42} + s_{51}$$

# Skew Schur functions and support



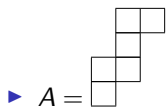
$$s_{A'} = s_{321} + s_{411} + 0s_{33} + s_{42}$$

$$s_{B'} = s_{321} + s_{411} + 0s_{33} + s_{42} + s_{51} \quad s_{C'} = s_{321} + s_{411} + s_{33} + 2s_{42} + s_{51}$$

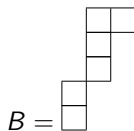


$$1 \not\leq 1 - 1, \quad 1 \leq 2 + 1 - 2$$

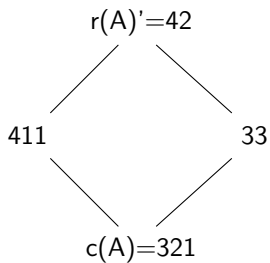
# Skew Schur functions and support



$$s_{A'} = s_{321} + s_{411} + s_{33} + s_{42}$$



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# Ribbon shapes and direct sums of ribbon shapes

## Definition

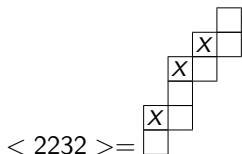
Let  $\alpha = (\alpha_1, \dots, \alpha_s)$  be a composition with  $\alpha_i \geq 2$ ,  $i \neq 1, s$ .

$R_\alpha$  denotes a skew-shape consisting of  $s$  column strips ( $1^{\alpha_i}$ ),  $i = 1, \dots, s$ , right to left, where any two of them overlap at most in one row.

Let  $0 \leq p < s$  be the number rows of size two. When  $p = s - 1$ ,  $R_\alpha$  is a ribbon and one writes  $R_\alpha = \langle \alpha \rangle$ . Otherwise, it is a direct sum of  $s - p$  ribbons.

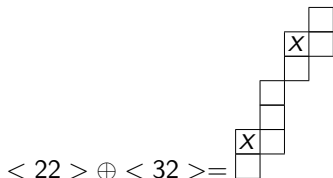
$\text{supp } R'_\alpha \subseteq [\alpha^+; (|\alpha| - p, p)]$ ,  $\alpha^+ = (\alpha_1^+, \dots, \alpha_s^+)$  the decreasing rearrangement of  $\alpha$ .

$R_{(2,2,3,2)}$



$p = 3,$

$[\alpha^+ = 32^3; 63]$



$p = 2$

$[\alpha^+ = 32^3; 72]$

# Ribbon shapes and direct sums of ribbon shapes

## Definition

Given the composition  $\alpha = (\alpha_1, \dots, \alpha_s)$ ,  $\alpha_i \geq 2$ ,  $i \neq 1, s$ , and a skew shape  $R_\alpha$ , let

$$R_\alpha^1 := R_\alpha, \text{ and } R_\alpha^{i+1} := R_\alpha^i \setminus \langle \alpha_i^+ \rangle, \quad i = 1, \dots, s-2,$$

giving priority to the rightmost column strip  $\langle \alpha_i^+ \rangle$  of  $R_\alpha$ , in case of equal size.

The *overlapping sequence* of  $R_\alpha$  is the non increasing sequence of nonnegative integers  $p_1 = p, p_2, \dots, p_{s-1}$ , where  $p_i$  is the number of rows with size two of  $R_\alpha^i$ ,  $1 \leq i \leq s-1$ . Note that  $0 \leq p_{i+1} \leq p_i \leq s-i$ , for  $i = 1, \dots, s-2$ .

$\langle 2232 \rangle =$

$p_1 = 3, p_2 = 1, p_3 = 0$   
 $\text{supp} R' \subseteq [32^3; 63]$

$\langle 1, 2, 2 \rangle \oplus \langle 3 \rangle =$

$p_1 = 2 = p_2, p_3 = 0$   
 $\text{supp} R' \subseteq [\alpha^+ = 32^21; 62]$



# Support criterion for a direct sum of a class of ribbons

## Theorem

Given the composition  $\alpha = (\alpha_1, \dots, \alpha_s)$  with  $\alpha_i > 1$ ,  $i \neq 1, s$ , consider  $R_\alpha$  with overlapping sequence  $(p_1, \dots, p_s)$ , and  $\nu' \in [\alpha^+; (|\alpha| - p, p)]$ .

$$c_{R_\alpha}^{\nu'} > 0 \quad \text{if and only if} \quad \nu_i \leq \sum_{j=i}^s \alpha_j^+ - p_i, \quad i = 1, \dots, s-1.$$

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$$c_{R_\alpha}^{\nu'} > 0 \quad \text{if and only if} \quad \nu_i \leq \sum_{j=i}^s \alpha_j^+ - p_i, \quad i = 1, \dots, s-1.$$

Equivalently,  $c_{R_\alpha}^{\nu'} > 0$  if and only if, for all  $i \in \{1, \dots, s-1\}$ ,

$$0 \leq \epsilon_i \leq \sum_{j=i+1}^s \alpha_j^+ - p_i,$$

where  $\epsilon_i$  is the number of lifted boxes from the last  $s - i$  rows of  $\alpha^+$  to the row  $\alpha_i^+$ .

►  $R = \langle 662322 \rangle \succeq (6^2 3 2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21 - 5)$

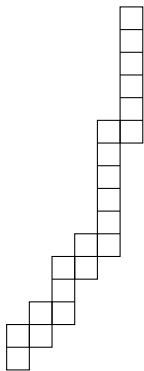
$$\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2 + 2 + 2 - 3 \quad p_3 = 3$$

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$$\epsilon_1 = 2 < \alpha_2^+ + \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 6 + 3 + 2 + 2 + 2 - 5 \quad p_1 = 5$$

$$\Rightarrow (876) \in \text{supp} R'$$

$$4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2 + 2 + 2 - 3 \Rightarrow (777) \notin \text{supp} R'$$



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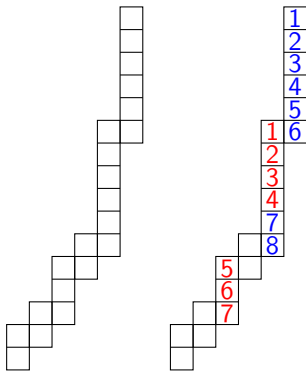
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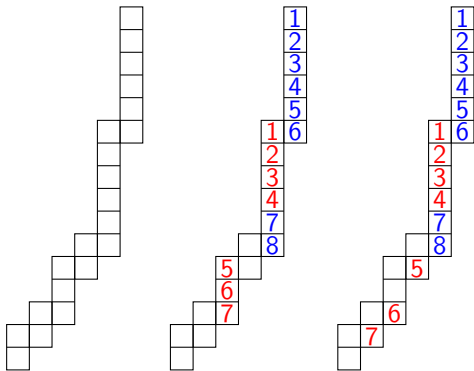
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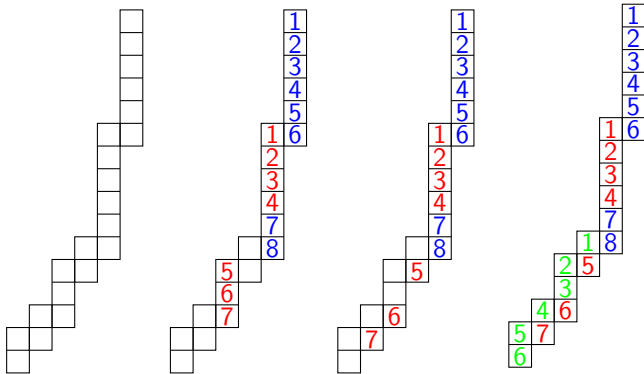
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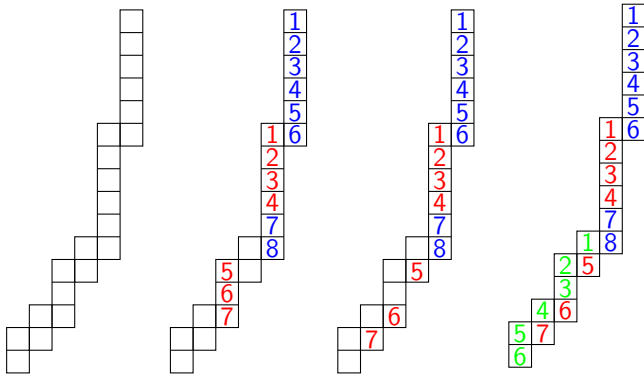
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$$\Rightarrow (876) \in \text{supp} R'$$

$$4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2 + 2 + 2 - 3 \Rightarrow (777) \notin \text{supp} R'$$



►  $R = \langle 662322 \rangle \quad (6^2 3 2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21 - 5)$

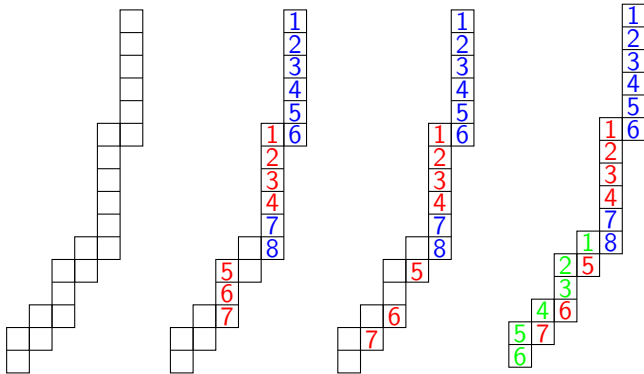
$$\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2 + 2 + 2 - 3 \quad p_3 = 3$$

$$\epsilon_2 = 1 < \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 3 + 2 + 2 + 2 - 3 \quad p_2 = 3$$

$$\epsilon_1 = 2 < \alpha_2^+ + \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 6 + 3 + 2 + 2 + 2 - 5 \quad p_1 = 5$$

$$\Rightarrow (876) \in \text{supp} R'$$

$$4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2 + 2 + 2 - 3 \Rightarrow (777) \notin \text{supp} R'$$





# Classification of a direct sum of a class of ribbon Schur functions with interval support

## Theorem

Given the composition  $\alpha = (\alpha_1, \dots, \alpha_s)$  with  $\alpha_i > 1$ ,  $i \neq 1, s$ , consider  $R_\alpha$  with overlapping sequence  $(p_1, \dots, p_s)$ .

$\text{supp} R'_\alpha \subsetneq [\alpha^+; (|\alpha| - p, p)]$  if and only if for some  $1 \leq i \leq s - 2$  with  $p_{i+1} \geq 1$ , there exist integers  $g_1, \dots, g_i \geq 0$  with  $\sum_{j=1}^i g_j \leq p_{i+1} - 1$ , such that

$$\alpha_j^+ + g_j \geq \sum_{q=i+1}^s \alpha_q^+ - p_{i+1} + 1, \quad j = 1, \dots, i.$$

In this case,

$(\alpha_1^+ + g_1, \dots, \alpha_i^+ + g_i, \sum_{q=i+1}^s \alpha_q^+ - p_{i+1} + 1, p_{i+1} - \sum_{j=1}^i g_j - 1)^+$  is not in the  $\text{supp} R'_\alpha$ .

## Theorem

Given the composition  $\alpha = (\alpha_1, \dots, \alpha_s)$  with  $\alpha_i > 1$ ,  $i \neq 1, s$ , consider  $R_\alpha$  with overlapping sequence  $(p_1, \dots, p_s)$ .

$c_{R_\alpha}^{\nu'} > 0$  whenever  $\nu' \in [\alpha^+; (|\alpha| - p, p)]$  if and only if for all  $1 \leq i \leq s - 2$  with  $p_{i+1} \geq 1$ , and for all integers  $g_1, \dots, g_i \geq 0$  with  $\sum_{j=1}^i g_j \leq p - 1$ , one has always, for some  $f \in \{1, \dots, i\}$ ,

$$\alpha_f^+ + g_f \leq \sum_{q=i+1}^s \alpha_q^+ - p.$$

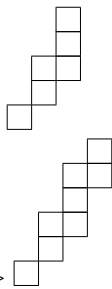
## Corollary

- ▶ If  $p_1 = 0$  or  $p_2 = 0$ ,  $\text{supp}R_\alpha = [\alpha^+; (|\alpha| - p, p)]$ .

$$\langle \alpha_1 \rangle \oplus \cdots \oplus \langle \alpha_s \rangle$$

$$\langle \alpha_1^+, \alpha_2^+ \rangle \oplus \cdots \oplus \langle \alpha_s \rangle$$

$$\langle \alpha_1^+, \alpha_2^+, \alpha_3^+ \rangle \oplus \cdots \oplus \langle \alpha_s \rangle$$



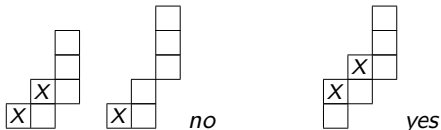
- ▶ If  $p_2 = 1, p_3 = 0$ ,  $R_\alpha$  has interval support **except** when

$$\alpha_1^+ \geq \sum_{q=2}^s \alpha_q^+$$

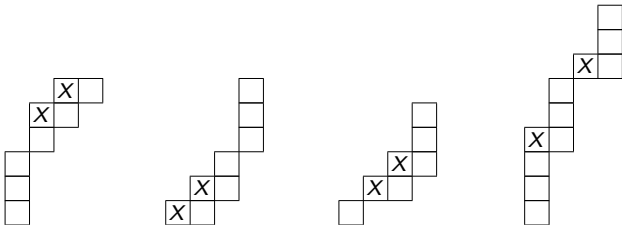
- ▶ If  $p_2 = 1 = p_3, p_4 = 0$ ,  $R_\alpha$  has interval support **except** when

$$\alpha_1^+ \geq \sum_{q=2}^s \alpha_q^+ \quad \text{or} \quad \alpha_1^+, \alpha_2^+ \geq \sum_{q=3}^s \alpha_q^+$$

$R_{(\alpha_1, \alpha_2, \alpha_3)}$  has interval support **except** when  $\alpha_1 \geq \alpha_2 + \alpha_3$  or  $\alpha_3 \geq \alpha_1 + \alpha_2$ .



$s = 4$



$p_1 = p_2 = 2$   
 $p_3 = 0$   
 no

$p_1 = p_2 = 2$   
 $p_3 = 1$   
 no

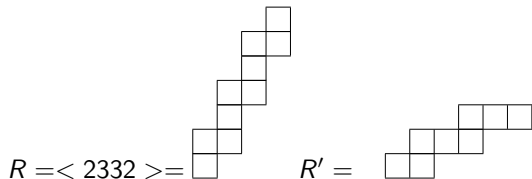
$p_1 = 2, p_2 = 1$   
 $p_3 = 0$   
 yes

$p_1 = 2, p_2 = 1$   
 $p_3 = 0$   
 yes

## Corollary

McNamara, van Willigenburg, 2011

Ribbon shapes whose column and row lengths differ at most one have full support.



## More examples

- $\alpha = (6, 2, 2, 2, 2, 7, 6)$ ,  $\alpha^+ = (7, 6, 6, 2, 2, 2, 2)$ ,  $i = 3$ ,  $p_4 = 3$

$$\begin{aligned} 7, 6 &\geq 2 + 2 + 2 + 2 - 2 \\ g_1 + g_2 + g_3 &= 0 \end{aligned} .$$

$$\nu = (7, 6, 6, 6, 2) \notin \text{supp} R'_\alpha$$

$$\begin{aligned} \nu &= (7, 6 + 1, 6 + 1, 2 + 2 + 2 + 2 - 2) = (7, 7, 7, 6) \notin \text{supp} R'_\alpha, \\ g_1 &= 0, g_2 = g_3 = 1 \end{aligned}$$

$$\begin{aligned} \nu &= (6 + 2, 7, 6, 2 + 2 + 2 + 2 - 2) = (8, 7, 6, 6) \notin \text{supp}(R'_\alpha), \\ g_1 &= 0 = g_2, g_3 = 2 \end{aligned}$$

$$\begin{aligned} \nu &= (7 + 2, 6, 6, 2 + 2 + 2 + 2 - 2) = (9, 6, 6, 6) \notin \text{supp}(R'_\alpha), \\ g_1 + g_2 + g_3 &= 2. \end{aligned}$$

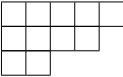
Note that  $p_4 = 3 \Rightarrow 3 \leq p_2, p_3 \leq 4$ .

If  $p_3 = 4$ ,  $7 + 3, 6 + 3 \not\geq 6 + 2 + 2 + 2 + 2 - 3$

II. Bad configurations. Full configurations.

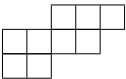
Classification of multiplicity free skew Schur functions with interval support.

# Definitions

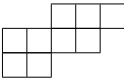
Example:  $\lambda = (5, 4, 2)$  = 



# Definitions

Example:  $\lambda/\mu = (5, 4, 2)/(2) =$  

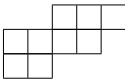
# Definitions

Example:  $\lambda/\mu = (5, 4, 2)/(2) =$  

$$s_{in} = (2, 2, 1, 3), \quad m^n - \text{shortness of } \mu \text{ is } 1$$

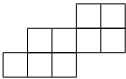
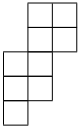

$$s_{out} = (2, 1, 2, 1, 1, 1), \quad m^n - \text{shortness of } \lambda^* \text{ is } 1$$

# Definitions

Example:  $\lambda/\mu = (5, 4, 2)/(2) =$  

$$s_{in} = (2, 2, 1, 3), \quad m^n - \text{shortness of } \mu \text{ is } 1$$

$$s_{out} = (2, 1, 2, 1, 1, 1), \quad m^n - \text{shortness of } \lambda^* \text{ is } 1$$

$$(\lambda/\mu)^\pi =$$
  ,  $(\lambda/\mu)' =$   ,  $(\lambda^*)^\pi =$  

The *sum*  $\lambda + \mu$  of two partitions  $\lambda$  and  $\mu$ , is the partition whose parts are equal to  $\lambda_i + \mu_i$ , with  $i = 1, \dots, \max\{\ell(\lambda), \ell(\mu)\}$ . Using conjugation, we define the *union*  $\lambda \cup \mu := (\lambda' + \mu')'$ . Equivalently,  $\lambda \cup \mu$  is obtained by taking all parts of  $\lambda$  jointly with those of  $\mu$  and rearranging all these parts in descending order.

### Example

Let  $\lambda = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array}$  and  $\mu = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}$ . Then,  $\lambda + \mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}$ ,  $\lambda \cup \mu = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array}$ .

Fix a positive integer  $n$ , and let  $\lambda$  and  $\mu$  be two partitions with length  $\leq n$ . The product  $\lambda^\pi \bullet_n \mu$  of two partitions  $\lambda$  and  $\mu$  is defined as

$$(\lambda_1 + \mu_1, \dots, \lambda_n + \mu_n)/(\lambda^*)^\pi,$$

where  $\lambda^* = \lambda_1^n / \lambda$ .

### Example

If  $\lambda = (3, 2^2, 0^2)$  and  $\mu = (2, 1^2, 0^2)$ , we obtain  
 $\lambda^\pi \bullet_5 \mu = (5, 4^3, 3^2)/(3^2, 1^2),$

$$\lambda^\pi \bullet \mu = \begin{array}{cc} & \square\square \\ & \square \\ \square & \square\square \\ \square & \square \\ \square & \square \end{array} .$$

# Problems

*If  $A$  is a skew shape with two or more components and  $A$  has interval support, then the components of  $A$  are ribbon shapes.*

# Problems

*If  $A$  is a skew shape with two or more components and  $A$  has interval support, then the components of  $A$  are ribbon shapes.*

When do the Schur function products have interval support?

Under what conditions do we have  $c'_A = 1$  whenever  $\nu' \in [c(A), r(A)']$ ?  
(Equivalently, when is it the case that  $s_A$  can be expressed as  $\sum_{\nu'} s_{\nu}$  where the sum runs over the interval  $[c(A), r(A)']$  in dominance order?)

When are the Schur function products multiplicity free and with interval support?

Is there any representation-theoretic or geometric explanation?

# Bad configurations

## Lemma


Let  $\xi \in [c(A), r(A)' = (n_1, \dots, n_s)]$  where  $\xi = (n_1) \cup \xi^1$ . Then

$$\xi^1 \notin \text{supp}(A \setminus V_1)' \Rightarrow \xi \notin \text{supp}A'.$$

## Proposition

Let  $A$  be a skew diagram with two or more connected components. If there is a component containing a two by two block of boxes, then  $A$  has no interval support.

## Example

The support of  $A =$  is not an interval.

$[c(A), r(A)'] = \{c(A) = 221, \xi = 311, r(A)' = 32\}$  with  $\xi \notin \text{supp}A' = \{c(A), r(A)'\}$ .



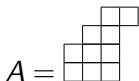
## Corollary

If  $A$  is a skew diagram with two or more components and the support of  $A$  is an interval, then the connected components of  $A$  are ribbon shapes.

## Corollary

Let  $A$  be a skew diagram such that  $\ell(c(A)) > \ell(r(A)') = s$  (equivalently, it has no block of maximal width), and the vertical strip  $V_s$  is a column of  $A$  of length greater than, or equal to 2. Then, the support of  $A$  is not an interval.

## Example



We have  $c(A) = (4, 3, 2, 1) \preceq \xi = (4, 4, 1, 1) \preceq r(A)' = (4, 4, 2)$ ,  
but  $\xi \notin \text{supp}A'$ .

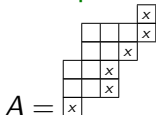
## Proposition

Let  $A$  be a connected skew diagram such that

$$\sigma^1 = (n_1, \bar{w}_2, \dots, \bar{w}_l, w_{l+1}, \dots, w_r) \in [c(A) = (w_1, \dots, w_r); r(A)' = (n_1, \dots, n_s)]$$

for some  $3 \leq l \leq r$  such that  $\bar{w}_k \leq w_k$  for  $k = 1, \dots, l$  and  $\bar{w}_l < w_l$ . Moreover, assume the existence of two integers  $2 \leq i < j \leq l$  such that  $\bar{w}_i \geq \bar{w}_j + 2$  and  $w_j > \bar{w}_j$ . Then the support of  $A$  is not an interval.

## Example



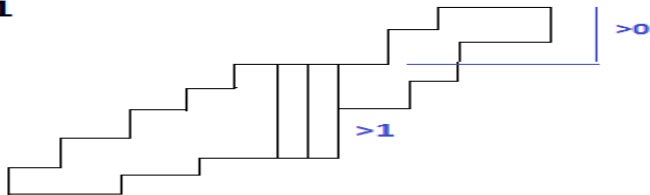
$$c(A) = (4, 4, 3, 2, 2)$$

$$r(A)' = (6, 4, 4, 1)$$

$$\sigma^1 = (6, \underline{4}, 2, 2, \underline{1})$$

$\text{supp} A' \subsetneq [c(A), r(A)']$ ,  $\xi = (5, 3, 3, 2, 2) \notin \text{supp} A'$  but  $c(A) \preceq \xi \preceq r(A)'$

**F1**



### Theorem

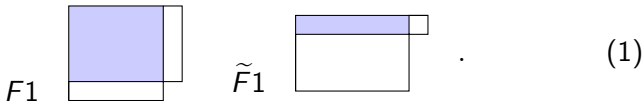
If, up to a  $\pi$ -rotation and/or conjugation,  $A$  is an  $F1$  configuration then  $\text{supp}(A') \subsetneq [c(A), r(A)']$ .



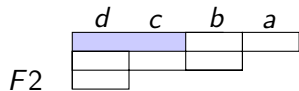
## Proposition

- ▶ (A. 99'; C. Bessenrodt and A. Kleshchev 99')  
 $c(\lambda/\mu) = r(\lambda/\mu)'$  if and only if  $\lambda/\mu = \nu$  or  $\lambda/\mu = \nu^\pi$ .
- ▶ (van Willigenburg 04')  $s_{\lambda/\mu} = s_\nu$  if and only if  $\lambda/\mu = \nu$  or  $\lambda/\mu = \nu^\pi$ .
- ▶ (A. 99') Let  $A$  be a skew diagram with  $c(A) \not\cong r(A)'$ . Then,  $\text{supp}A = \{c(A), r(A)'\}$  if and only if, up to a  $\pi$ -rotation/ or conjugation and up to a block of maximal width or maximal depth,  $A$  either is an  $F1$  or an  $\tilde{F}1$  configuration.

$F1 = ((a+1)^x, a)/(a^x)$ , and  $\tilde{F}1 = (a+1, a^x)/(a)$ ,  $a, x \geq 1$ :



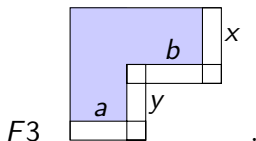
Consider skew diagram  $\lambda/\mu$  with three rows, where  $\mu = (d + c)$  is one row rectangle and  $\lambda^* = (a + b + c, a)$  is a fat hook, or vice versa, for some integers  $a, d \geq 1$  and  $b, c \geq 0$ :



### Proposition

Let  $\lambda/\mu$  be the skew diagram  $F2$ . Then, the support of  $\lambda/\mu$  is an interval if and only if it is  $a \leq c + 1$  and  $d \leq b + 1$ . In this case we say that  $\lambda/\mu$  is an  $A2$  configuration.

Consider the ribbon skew diagram  $\lambda/\mu$ , with  $\mu = ((a+b+1)^x, a^y)$  and  $\lambda^* = ((b+1)^{y+1})$ , for some integers  $a, b, x, y \geq 1$ , as illustrated by:

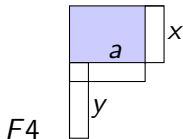


### Proposition

Let  $\lambda/\mu$  be a  $F3$  configuration. Then its support is an interval if and only if  $a = x = 1$ , or  $a = 1$  and  $x \leq y + 1$ , or  $a \leq b + 1$  and  $x = 1$ , in which case it is called an  $A3$  configuration.



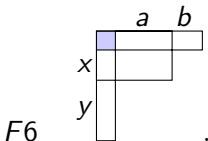
Consider the skew diagram  $\lambda/\mu$  of type  $F4$  defined by the partitions  $\lambda = ((a+2)^x, a+1, 1^y)$  and  $\mu = ((a+1)^x)$  for some  $a, x, y \geq 1$  such that not both  $x$  and  $y$  are equal to 1:



### Proposition

If  $\lambda/\mu$  is the skew diagram  $F4$ , then its support is an interval if and only if  $a = 1$  and  $x \leq y + 1$ , or  $a \geq 2$  and  $x = 1$ , in which case it is called an  $A4$  configuration.

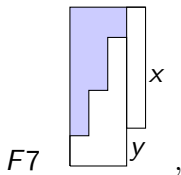
The next family of skew diagrams  $\lambda/\mu$  is designated by  $F6$  and it is defined by partitions  $\lambda = (a + b + 1, (a + 1)^x, 1^y)$  and  $\mu = (1)$ , for some integers  $a, x > 0$  and  $b, y \geq 1$ :



### Proposition

The support of the skew diagram  $F6$  is an interval if and only if  $b = y = 1$ . In this case it is called an  $A6$  configuration.

Finally, consider the skew diagram  $A$ , with  $f + 1 \geq 4$  columns and  $x + y$  rows,  $x, y \geq 1$ , having the form



where

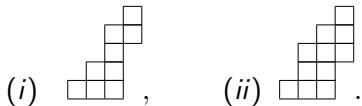
- ▶ the first  $f$  columns end in the same row and have pairwise distinct lengths,
- ▶  $x$  is the length of the  $(f + 1)$ th column,
- ▶ the  $f$ th column starts at least one box below the starting row of the  $(f + 1)$ th column, and
- ▶ the first column has length  $\leq y$ .

## Proposition

Let  $A$  be an  $F7$  configuration. Denote by  $(w_1, \dots, w_f)$  the partition formed by the first  $f$  columns of  $A$ , and let  $k \geq 0$  be the number of rows shared by the  $f$ th and  $(f+1)$ th columns of  $A$ . The support of  $A$  is an interval if and only if  $f = 3$ ,  $w_1 = 1$ ,  $x = w_2$  and  $k = w_2 - 1$ . In this case the diagram is called an  $A7$  configuration.

## Example

Some  $A7$  configurations are shown below:



## Theorem (Gutschwager '06, Thomas and Yong '05, King et al '09)

The basic skew Schur function  $s_{\lambda/\mu}$  is multiplicity-free if and only if one or more of the following is true:

*R0*  $\mu$  or  $\lambda^*$  is the zero partition 0;

*R1*  $\mu$  or  $\lambda^*$  is a rectangle of  $m^n$ -shortness 1;

*R2*  $\mu$  is a rectangle of  $m^n$ -shortness 2 and  $\lambda^*$  is a fat hook;

*R3*  $\mu$  is a rectangle and  $\lambda^*$  is a fat hook of  $m^n$ -shortness 1;

*R4*  $\mu$  and  $\lambda^*$  are rectangles.

## Theorem (Gutschwager '06, Thomas and Yong '05, King et al '09)

The basic skew Schur function  $s_{\lambda/\mu}$  is multiplicity-free if and only if one or more of the following is true:

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*R3*  $\mu$  is a rectangle and  $\lambda^*$  is a fat hook of  $m^n$ -shortness 1;

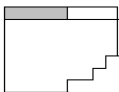
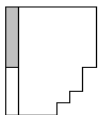
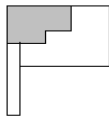
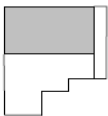
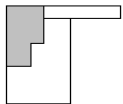
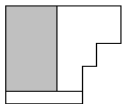
*R4*  $\mu$  and  $\lambda^*$  are rectangles.

## Corollary (Stembridge '00)

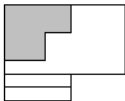
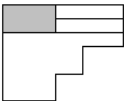
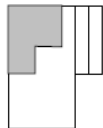
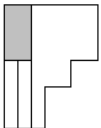
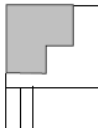
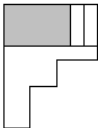
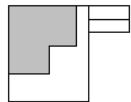
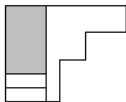
The product  $s_\mu s_\nu$  is multiplicity-free if and only if

- (i)  $\mu$  or  $\nu$  is a one-line rectangle, or
- (ii)  $\mu$  is a two line rectangle and  $\nu$  is a fat hook, or
- (iii)  $\mu$  is a rectangle and  $\nu$  is a near rectangle, or
- (iv)  $\mu$  and  $\nu$  are rectangles.

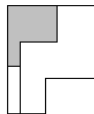
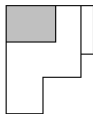
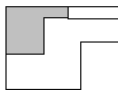
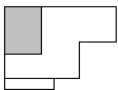
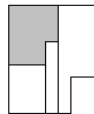
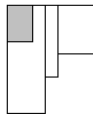
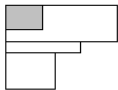
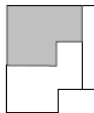
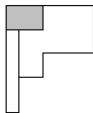
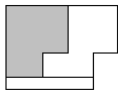
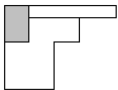
**R1**



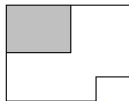
**R2**



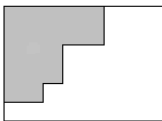
**R3**



**R4**



**R0**

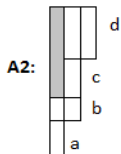




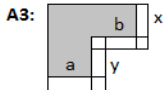
## Theorem

The basic skew Schur function  $s_{\lambda/\mu}$  is multiplicity-free and has interval support if and only if, up to a block of maximal width or maximal length, and up to a  $\pi$ -rotation and/or conjugation, one or more of the following is true:

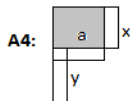
- (i)  $\mu$  or  $\lambda^*$  is the zero partition 0.
- (ii)  $\lambda/\mu$  is a two column or a two row diagram.
- (iii)  $\lambda/\mu$  is an A2, A3, A4, A6 or A7 configuration.



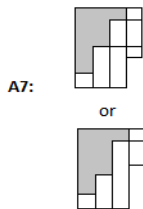
$$a \leq c+1 \text{ and} \\ b \leq d+1$$



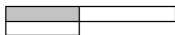
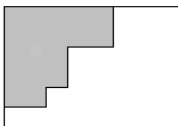
$$a=x=1, \text{ or} \\ a=1 \text{ and } x \leq y+1, \text{ or} \\ x=1 \text{ and } a \leq b+1$$



$$a=1 \text{ and } x \leq y+1, \text{ or} \\ a>1 \text{ and } x=1$$



with  $\ell(c_2) = \ell(c_4)$



A7:

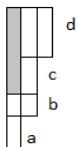


or



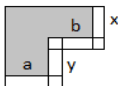
with  $\ell(c2) = \ell(c4)$

A2:



$a \leq c+1$  and  $b \leq d+1$

A3:



$a=x=1$ , or  
 $a=1$  and  $x \leq y+1$ , or  
 $x=1$  and  $a \leq b+1$

A4:



$a=1$  and  $x \leq y+1$ , or  
 $a > 1$  and  $x=1$

A6:



## Corollary

The Schur function product  $s_\mu s_\nu$  is multiplicity-free and has interval support if and only if one or more of the following is true:

- (a)  $\mu$  or  $\nu$  is the zero partition.
- (b)  $\mu$  and  $\nu$  are both rows or both columns.
- (c)  $\mu = (1^x)$  is a one-column rectangle and  $\nu = (a, 1^y)$  is a hook such that either  $a = 2$  and  $1 \leq x \leq y + 1$ , or  $a \geq 3$  and  $x = 1$  (or vice versa).
- (c')  $\mu = (x)$  is a one-row rectangle and  $\nu = (z, 1^a)$  is a hook such that either  $a = 1$  and  $1 \leq x \leq z$ , or  $a \geq 2$  and  $x = 1$  (or vice versa).

## Corollary

The Schur function product  $s_\mu s_\nu$  has interval support if and only if one of the conditions above or one of the following is true:

$\mu = (r_1, 1^{r_2})$  and  $\nu = (s_1, 1^{s_2})$  are hooks such that  $s_2 = r_2 = 1$ , and either  $r_1 = s_1 \geq 2$  or  $r_1 = 2$ ,  $s_1 = r_1 + 1$  (or vice versa).

