Some Positivity Questions for Coxeter Groups

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Eulerian Posets and the **cd**-index

- Flags in graded and Eulerian posets
- The **cd**-index for Eulerian posets
- Inequalities for flags in polytopes and spheres

Coxeter Groups and the Bruhat order

- Reflection orderings and chain enumeration
- The complete **cd**-index of a Coxeter group
- Connection to Kazhdan-Lusztig and g-polynomials

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- For Eulerian posets, only Fibonacci many f_S are linearly independent (via generalized Dehn-Sommerville equations).

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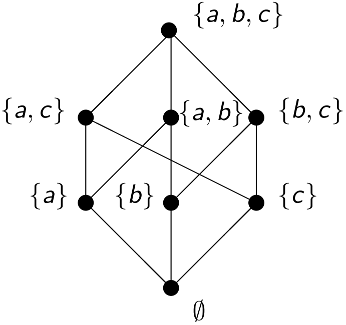

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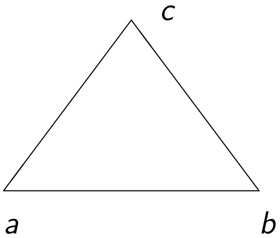

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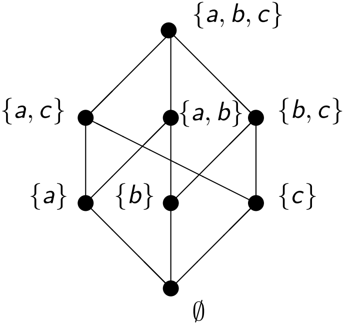

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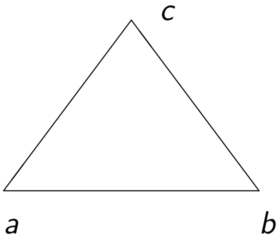

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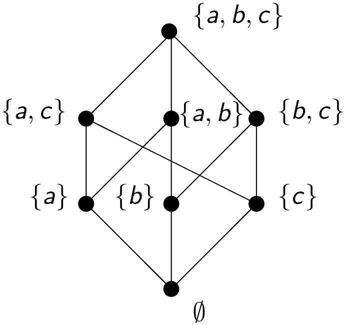

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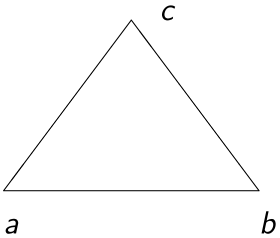

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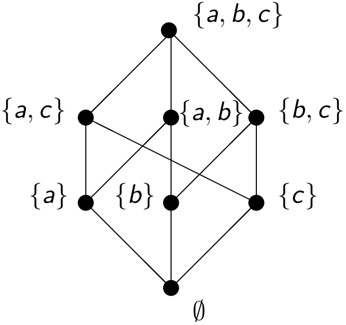

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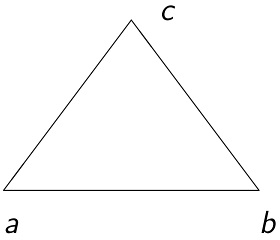

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$$\begin{aligned}\Psi_P &= \mathbf{aa} + 2\mathbf{ba} + 2\mathbf{ab} + \mathbf{bb} \\ &= (\mathbf{a}+\mathbf{b})^2 + (\mathbf{ab}+\mathbf{ba}) \\ &= \mathbf{c}^2 + \mathbf{d} = \Phi_P\end{aligned}$$

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- Karu & Ehrenborg: The simplex (Boolean lattice) minimizes termwise among all Gorenstein* *lattices* of the same rank.

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If k is minimal among all such expressions for v, then $s_1 s_2 \cdots s_k$ is called a reduced expression for v and $k = l(v)$ is called the length of v.

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$m(s_i, s_j) = 2$ otherwise if $i < j$ (and so $s_i s_j = s_j s_i$).

Bruhat order on (W, S)

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We will extend this to a nonhomogeneous **cd**-polynomial $\widetilde{\Phi}_{u,v}$ whose coefficients $[w]_{u,v}$ can be used to compute the Kazhdan-Lusztig polynomial of the interval $[u, v]$ (an analog of the g-polynomial).

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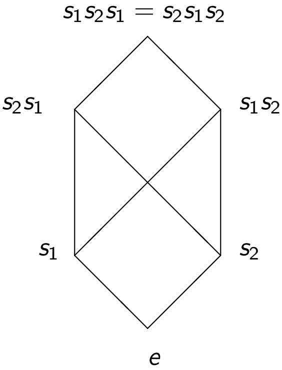

$$S_3 = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 = s_2 s_1 s_2\}$$
$$= \{e, (1, 2), (2, 3), (1, 2, 3), (1, 3, 2), (1, 3)\}$$

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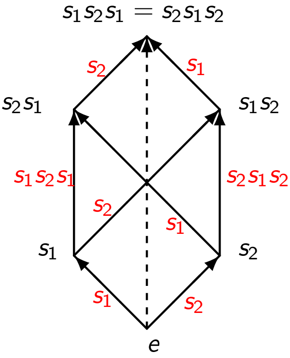
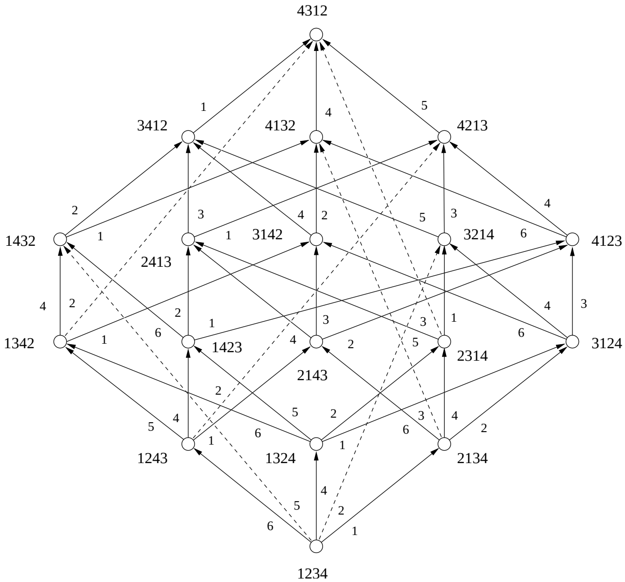


Bruhat graph of $[123, 321]$ in $W = S_3$

Bruhat graph of $[1234, 4312]$ in $W = S_4$



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Dyer: Reflection orderings exist for all Coxeter groups.

Reflection ordering on W

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For any dihedral reflection subgroup (W', S'), where $S' = \{a, b\}$, either

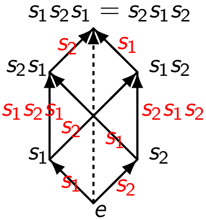
$$a <_T aba <_T ababa <_T \cdots <_T babab <_T bab <_T b$$

or

$$b <_T bab <_T babab <_T \cdots <_T ababa <_T aba <_T a$$

Dyer: Reflection orderings exist for all Coxeter groups.

In S_3, $s_1 = (1,2) <_T s_1s_2s_1 = s_2s_1s_2 = (1,3) <_T s_2 = (2,3)$ is a reflection ordering.



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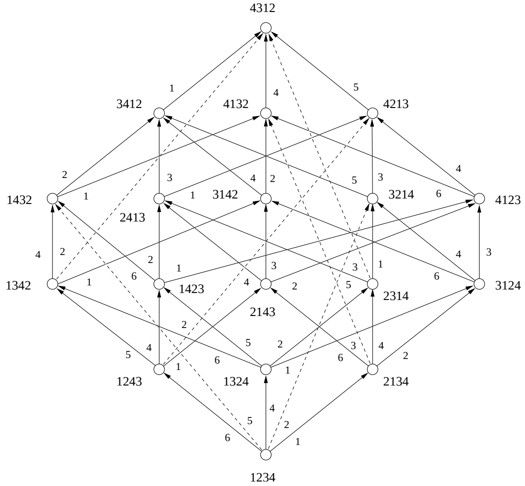
$$(12) <_T (13) <_T (14) <_T (23) <_T (24) <_T (34)$$



Paths in the Bruhat graph

Directed $u - v$ paths in the Bruhat graph have lengths
$l(v) - l(u)$, $l(v) - l(u) - 2$, $l(v) - l(u) - 4$, $\ldots$

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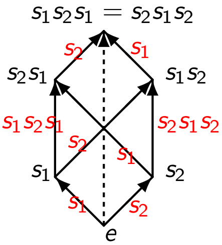

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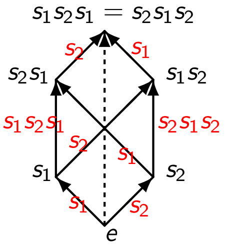

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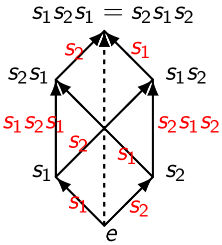

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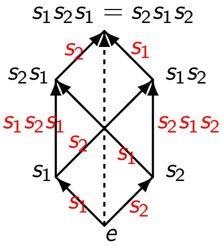

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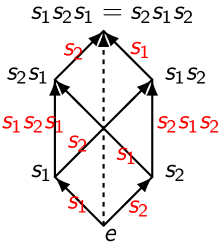

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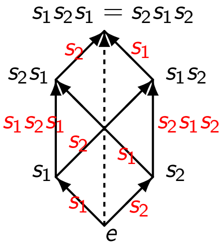

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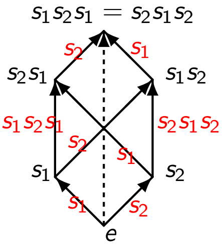

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Maximum-length paths are maximal chains in poset $[u, v]$

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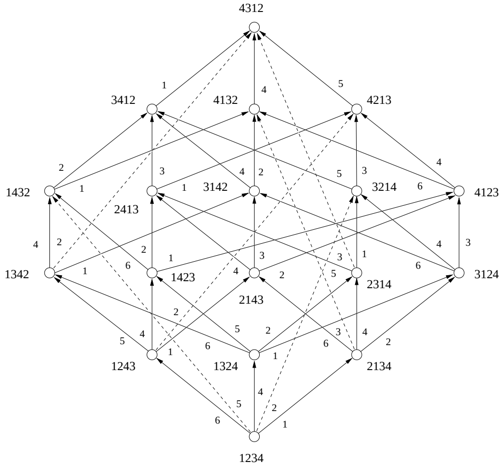
Here $b_3 = b_{21} = b_{12} = b_{111} = b_1 = 1$

Descent distribution of $[1234, 4312]$ in S_4



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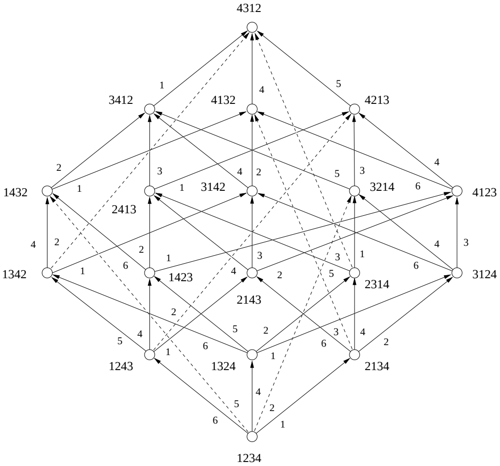

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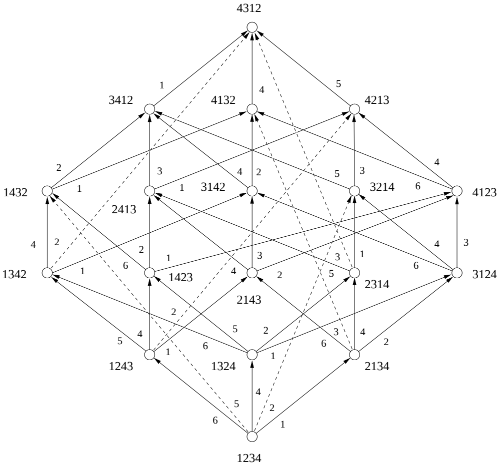

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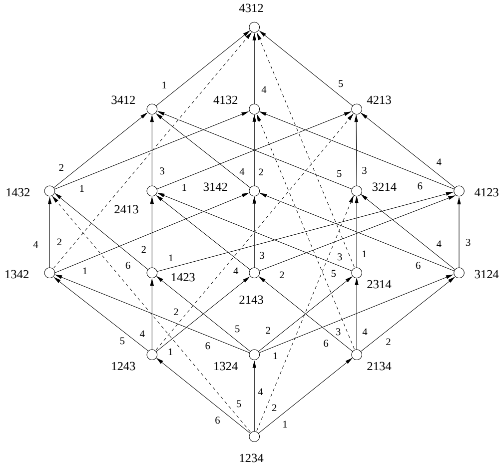

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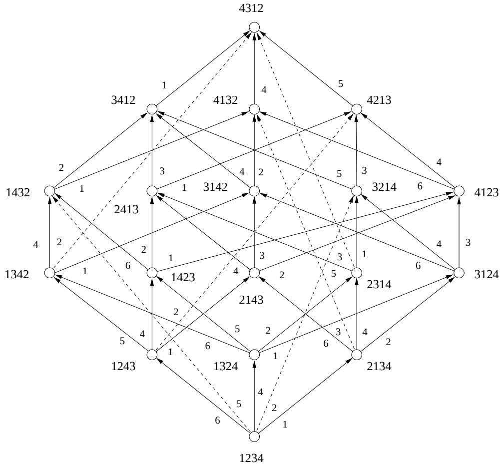

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The **ab**-index of $[u, v]$

Composition $\alpha = \alpha_1 \alpha_2 \cdots \alpha_k \longmapsto$

ab-word $w_\alpha = \mathbf{a}^{\alpha_1 - 1}\mathbf{b}\mathbf{a}^{\alpha_2 - 1}\mathbf{b} \cdots \mathbf{b}\mathbf{a}^{\alpha_k - 1}$

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This polynomial $\widetilde{\Phi}_{u,v}(\mathbf{c},\mathbf{d})$ is called the complete **cd**-index of $[u, v]$.

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The complete **cd**-index of $[u, v]$

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$$\widetilde{\Phi}_{u,v}(c, d) = \sum_{w} [w]_{u,v}\ w$$

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So, e.g., $[\mathbf{c}^2\mathbf{dc}]_{e,4321} = 2$

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There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$, such that, for all $u, v \in W$,

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Elias and Williamson, The Hodge theory of Soergel bimodules, *Annals of Mathematics* **180** (2014), 1089–1136.

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Note that if $P_{u,v} = p_0 + p_1 q + \cdots$ then $a_j(u,v) = \sum_{i=0}^{j} \binom{n-j-i}{j-i} p_i$, so $P_{u,v} \geq 0 \Longrightarrow a_i(u,v) \geq 0$, for all i, all $u < v$

Connection to g-polynomial of polytopes / Eulerian posets

First few a_i:

$$a_0 = [\mathbf{c}^n]$$

$$a_1 = [\mathbf{c}^{n-2}\mathbf{d}] + [\mathbf{c}^{n-2}]$$

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$$+ [\mathbf{c}^{n-6}\mathbf{d}^2] - [\mathbf{c}^{n-6}\mathbf{d}\mathbf{c}^2] + [\mathbf{c}^{n-6}\mathbf{d}] + [\mathbf{c}^{n-6}]$$

restrict to terms of top degree n only:

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get

$g([u,v]^*, q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \widehat{a}_i \; q^i \, B_{n-2i}(-q)$ (Bayer-Ehrenborg),

Example:

$W = S_4$, $u = 1234$ and $v = 4231$, lex reflection order

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so $(a_0, a_1, a_2) = (1, 4, 2)$.

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1) Since the top-degree **cd** terms constitute the ordinary **cd**-index of the (spherical) Eulerian poset $[u, v]$, it follows from the result of Karu that these coefficients are always nonnegative.

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Some References

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