q-analogues of Naruse's hook-length formula for skew shapes

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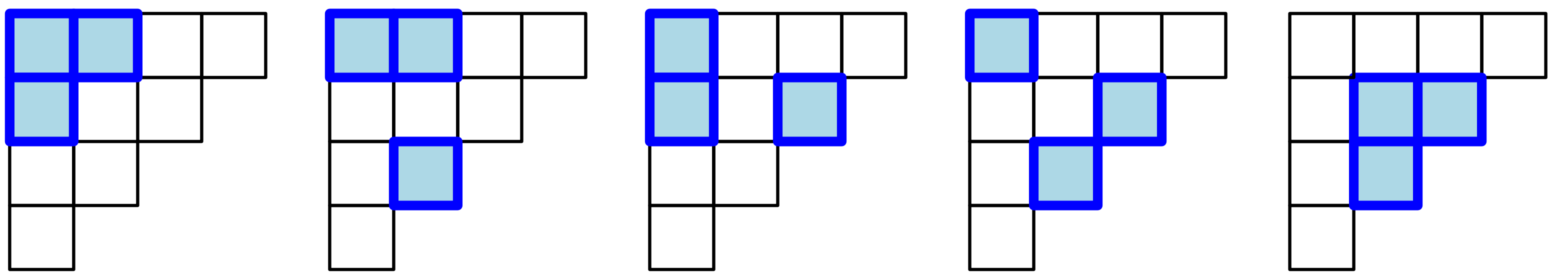
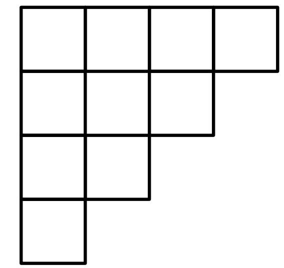


BIRS workshop Positivity in Algebraic Combinatorics
August 15, 2015

Notation

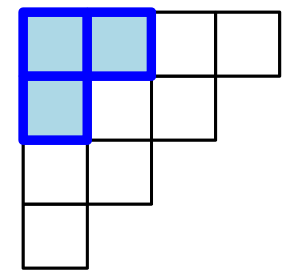
λ: partition shape

$\delta_5 =$

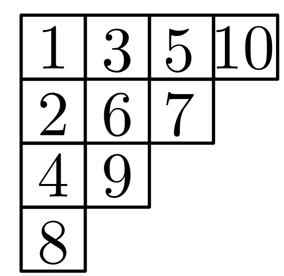


λ/μ: skew shape

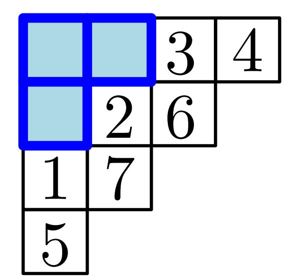
$\delta_5/\delta_3 =$



$f^{\lambda} = \#$ SYT of shape λ



$f^{\lambda/\mu} = \#$ SYT of shape λ/μ



Hook-length formula

Theorem (Frame-Robinson-Thrall 1954)

$$f^\lambda = |\lambda|! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)},$$

$h(i,j) = \lambda_i - i + \lambda'_j - j + 1$ is the **hook-length** of (i,j)

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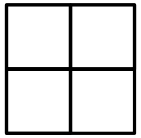
Example



$$f^{\boxplus} = \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right|$$

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Example

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

$$f^{\boxplus} = \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right| = \frac{4!}{3 \cdot 1 \cdot 2^2 \cdot 3} = 2$$

q-analogue hook-length formula

Theorem (Stanley 1971)

$$s_\lambda(1, q, q^2, \ldots) = q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1 - q^{h(i,j)}}, \qquad (*)$$

where $b(\lambda) = \sum_i (i-1)\lambda_i$

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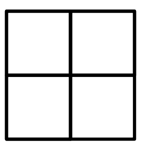

3	2
2	1

$$s_{\boxplus}(1, q, q^2, \ldots) = \frac{q^2}{(1-q^1)(1-q^2)^2(1-q^3)}$$

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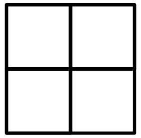

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$$s_{\boxplus}(1, q, q^2, \ldots) = \frac{q^2}{(1-q^1)(1-q^2)^2(1-q^3)}$$

- $(*)$ implies hook-length formula.

Outline

$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

$$s_\lambda(1, q, \ldots) = \frac{q^{b(\lambda)}}{\prod_{u\in\lambda}(1 - q^{h(u)})}$$

Outline

$$f^{\lambda} = \frac{|\lambda|!}{\prod_{u \in \lambda} h(u)}$$

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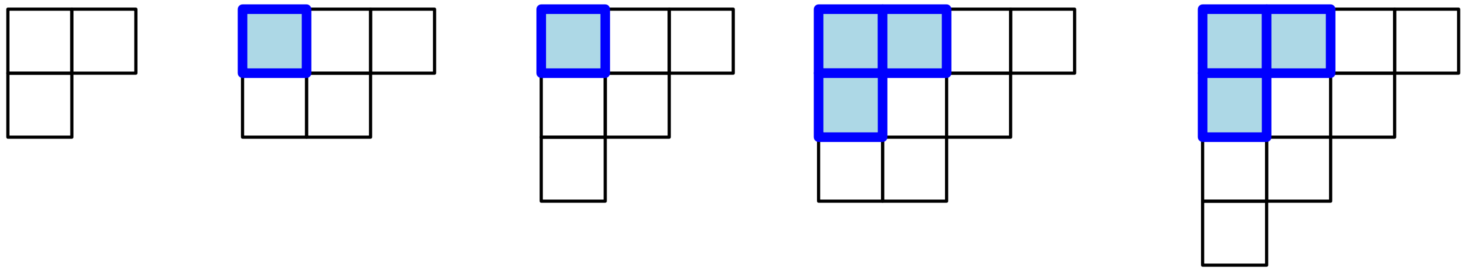
$$f^{\lambda/\mu} = ?$$

$$s_{\lambda/\mu}(1, q, \ldots) = ?$$

No product formula for $f^{\lambda/\mu}$

Example

$\lambda/\mu :$



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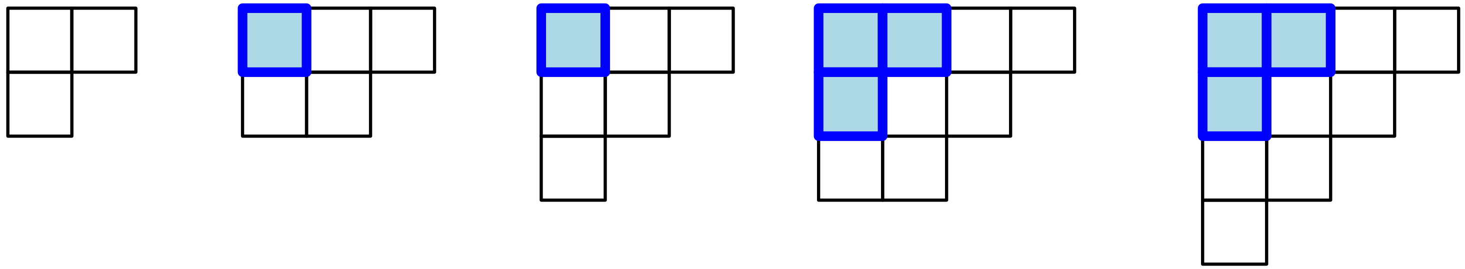
Example

λ/μ :



$f^{\lambda/\mu}$: $2 \quad 5 \quad 16 \quad 61 \quad 272 \quad \ldots$

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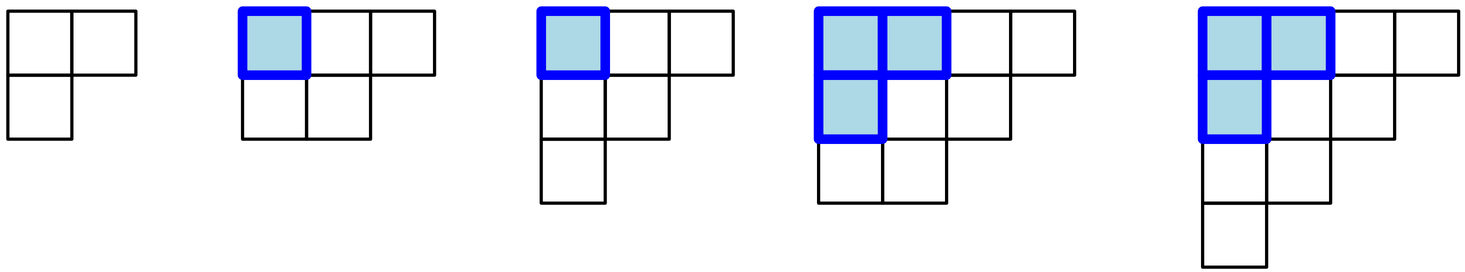
Example

λ/μ :



$f^{\lambda/\mu}$: 2 5 16 61 272 $\ldots$

Euler numbers E_n

$$E_{2n+1} = f^{\delta_{n+2}/\delta_n}$$

Alternating formulas for $f^{\lambda/\mu}$

Jacobi-Trudi formula

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{n}.$$

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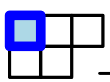
Example

f  $= 4! \cdot \det \begin{bmatrix} \frac{1}{2!} & \frac{1}{4!} \\ \frac{1}{1!} & \frac{1}{2!} \end{bmatrix}$

$$= 4! \cdot \left(\frac{1}{4} - \frac{1}{24} \right) = 5.$$

Positive formulas for $f^{\lambda/\mu}$

Littlewood-Richardson rule

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu},$$

where $c^{\lambda}_{\mu,\nu}$ are the **LR coefficients**.

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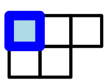
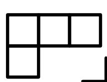
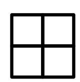
f  $= 1 \cdot f$  $+ 1 \cdot f$ 

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Example

$$f^{(3,2)/(1)} = 1 \cdot f^{(3,1)} + 1 \cdot f^{(2,2)}$$

$$= 1 \cdot 3 + 1 \cdot 2 = 5.$$

- other positive formula by Okounkov-Olshanski 1998.

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

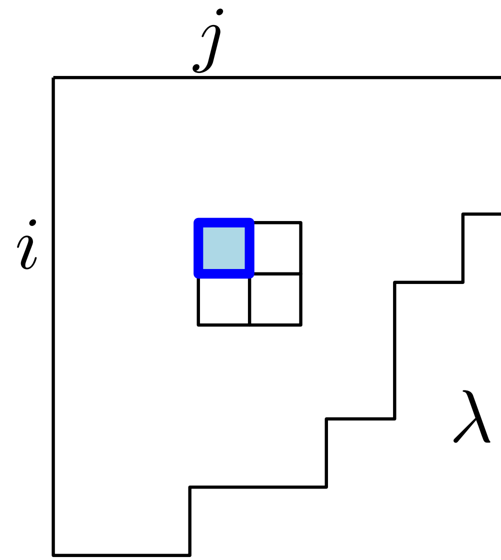
$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of **excited diagrams** of λ/μ.

Excited diagrams of λ/μ

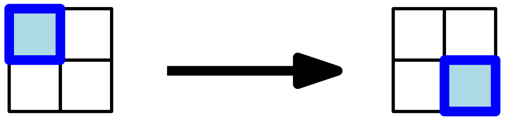
A cell $(i,j) \in S \subseteq \lambda$ is **excited** if

$$(i+1,j),(i,j+1),(i+1,j+1) \notin S.$$



An **excited move** on an excited cell (i,j):

replace (i,j) in S by $(i+1,j+1)$



Excited diagrams of λ/μ (cont.)

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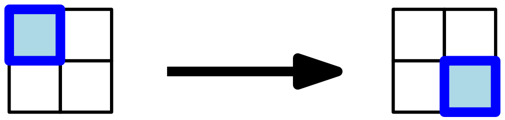

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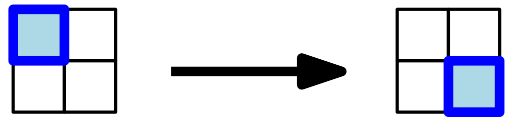

Definition: (Ikeda-Naruse 2009, Kreiman 2005)

Excited diagrams $\mathcal{E}(\lambda/\mu)$: diagrams obtained from μ by applying iteratively excited moves on active squares.

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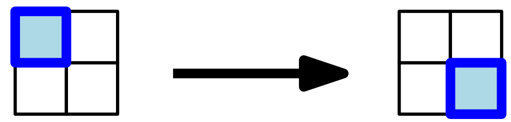

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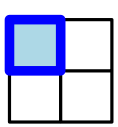
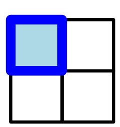
$\mathcal{E}($  $) = \{$  $\}$

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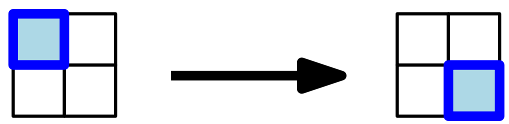

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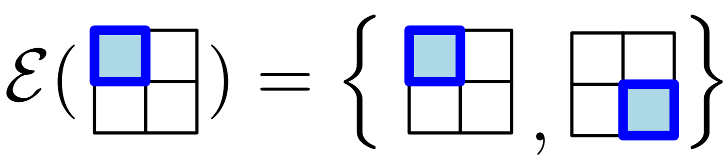

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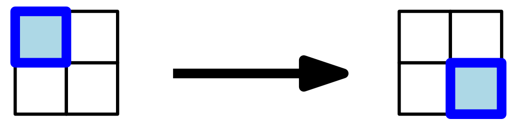

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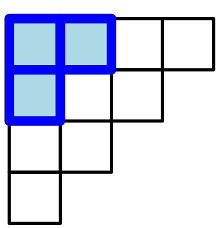
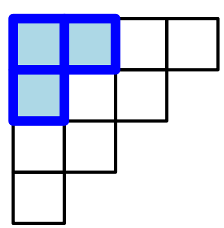
$\mathcal{E}($  $):$



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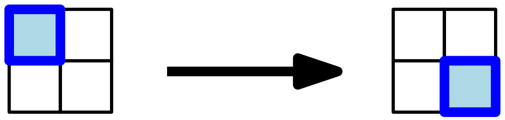

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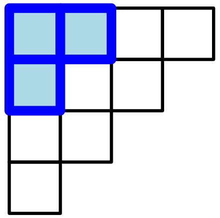
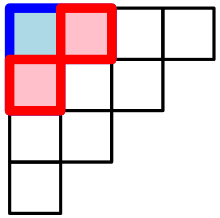
$\mathcal{E}($  $)$:



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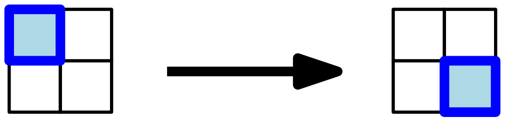

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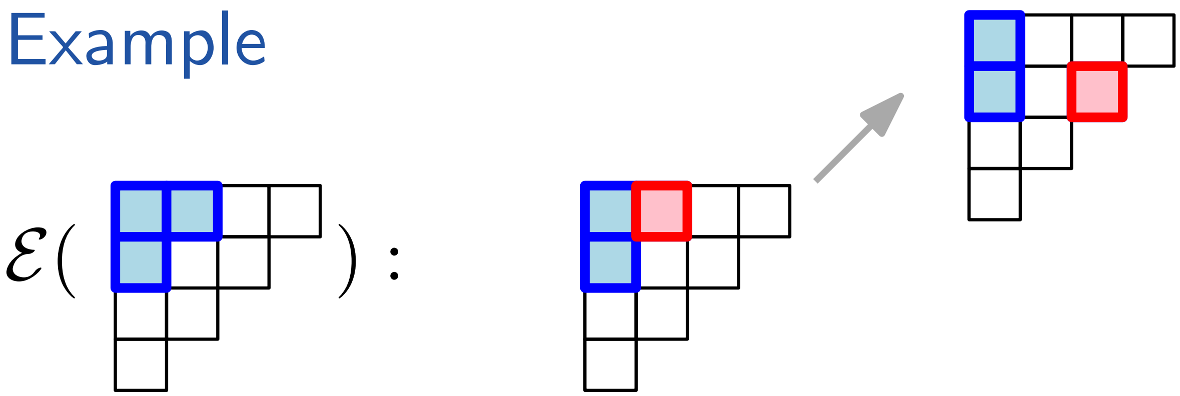

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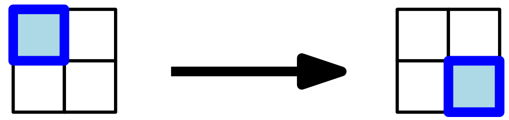

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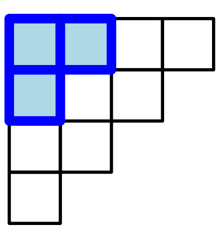
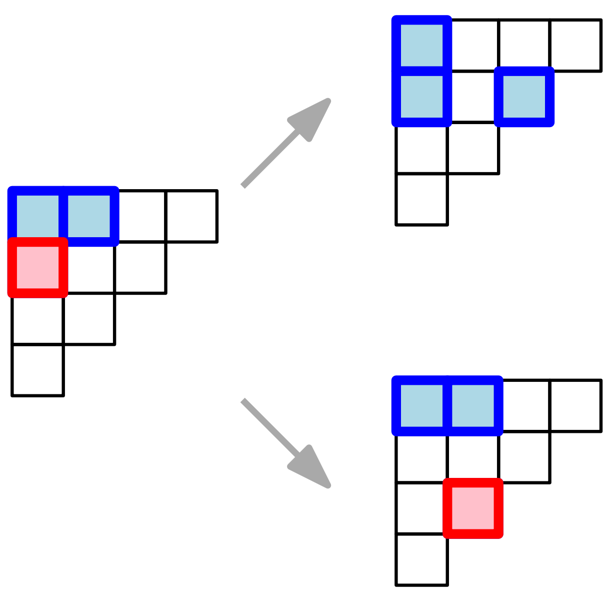
$\mathcal{E}($  $)$:



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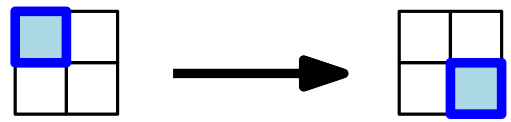

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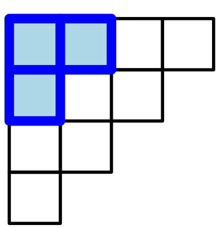
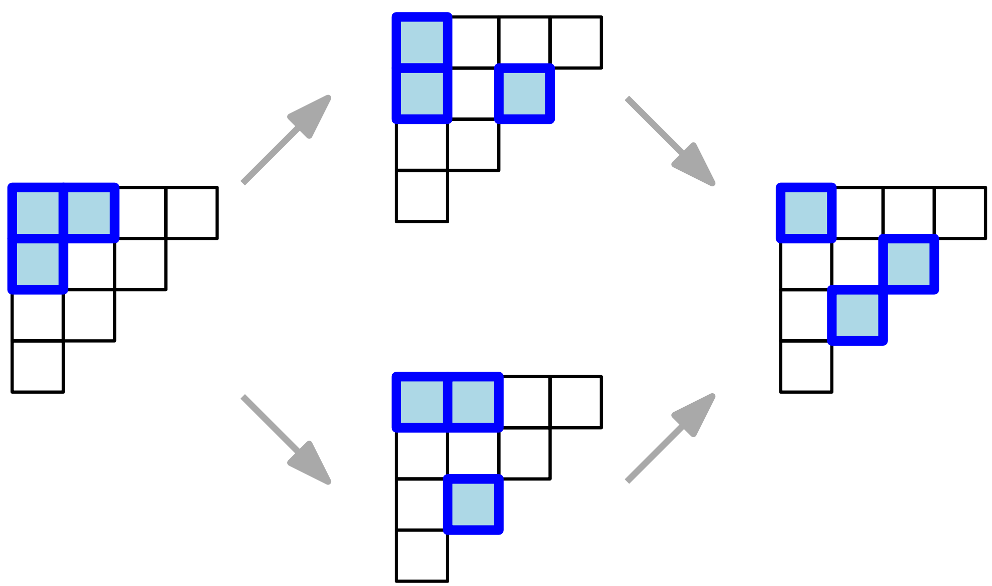
$\mathcal{E}($  $)$:



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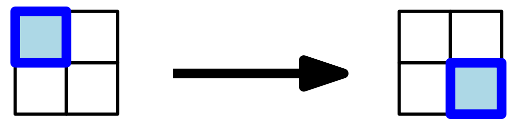

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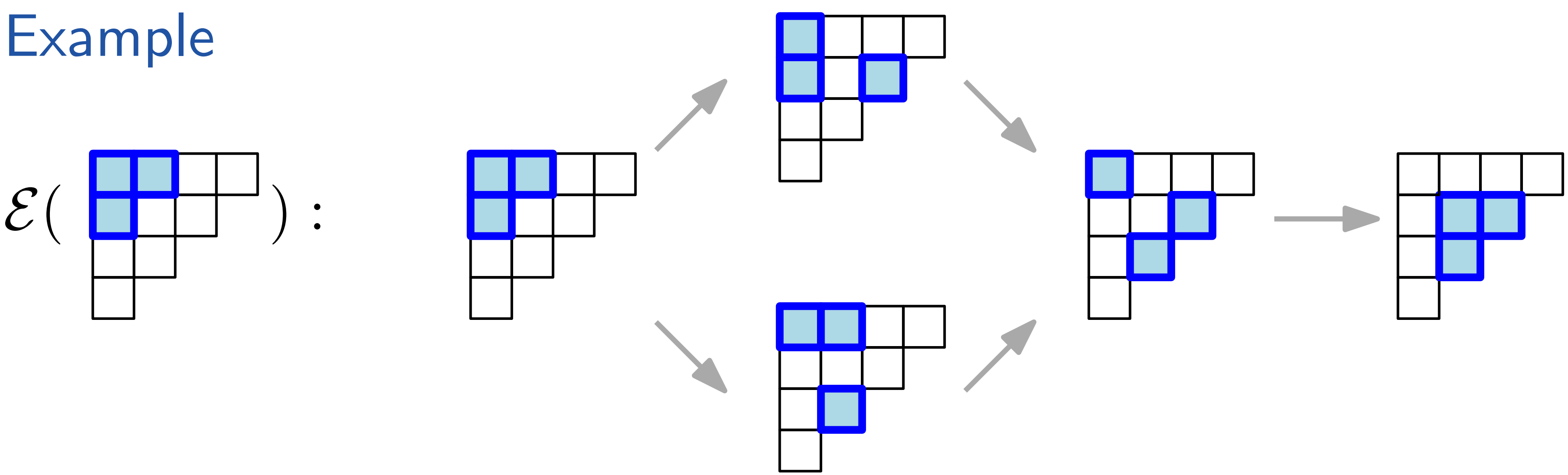

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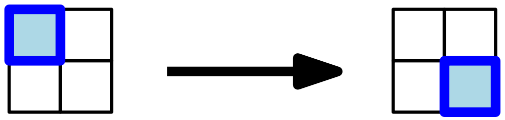

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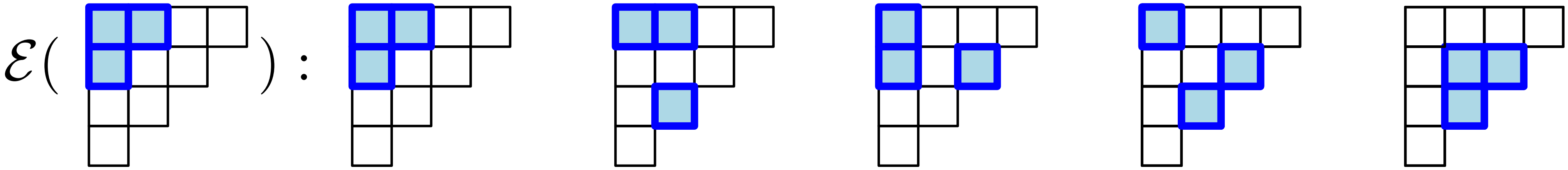

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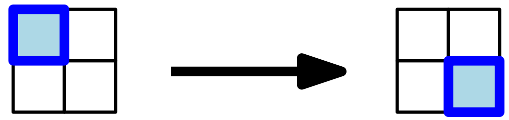

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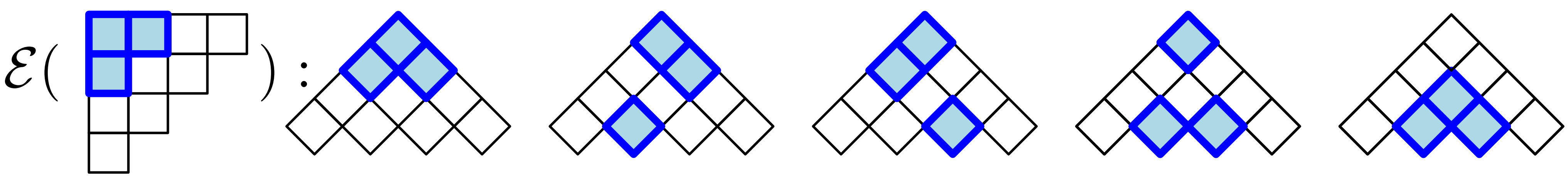

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

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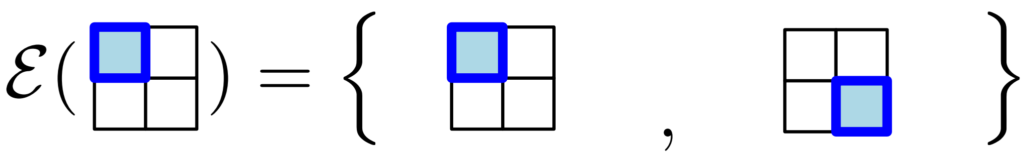
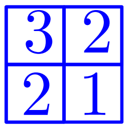
where $\mathcal{E}(\lambda/\mu)$ is the set of **excited diagrams** of λ/μ.

Example





Naruse's "hook-length" formula for $f^{\lambda/\mu}$

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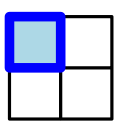
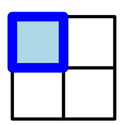
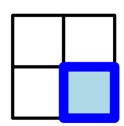
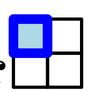
$\mathcal{E}($  $) = \{$  ,  $\}$

3	2
2	1

f  $= 3! \cdot \left(\frac{1}{1\cdot 2\cdot 2} + \frac{1}{2\cdot 2\cdot 3} \right)$

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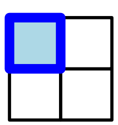
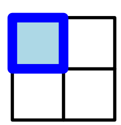
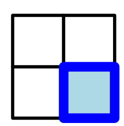
$\mathcal{E}($  $) = \{$  ,  $\}$

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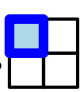
f  $= 3! \cdot \left(\frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right) = 3! \left(\frac{1}{4} + \frac{1}{12} \right) = 2.$

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

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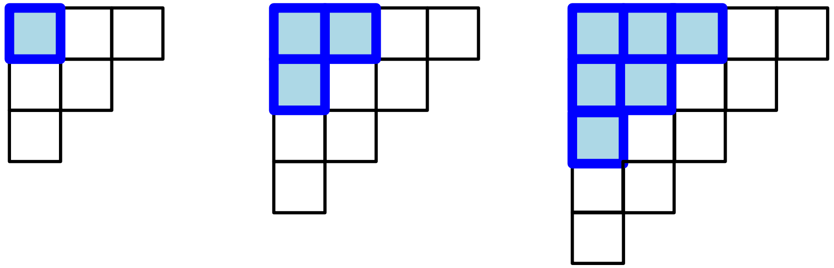
Example zigzag δ_{n+2}/δ_n:



$\cdots$

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

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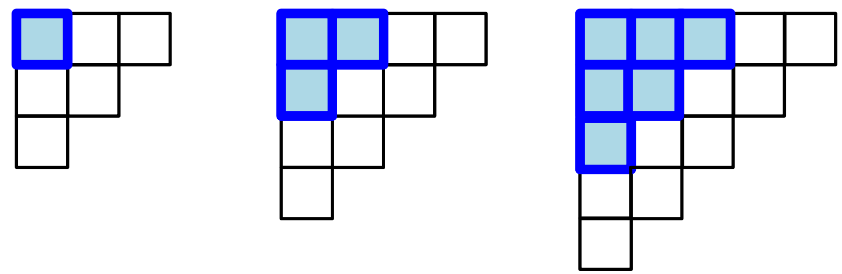

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$$E_{2n+1} = (2n+1)! \sum_{\mathsf{p} \in \mathsf{Dyck}(n)} \prod_{(a,b) \in \mathsf{p}} \frac{1}{2b+1},$$

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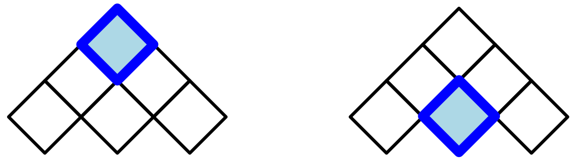
Example



$$E_{2n+1} = (2n+1)! \sum_{\mathsf{p} \in \mathsf{Dyck}(n)} \prod_{(a,b) \in \mathsf{p}} \frac{1}{2b+1},$$

$$16 = E_5 = 5! \cdot \left(\frac{1}{3 \cdot 3} + \frac{1}{3 \cdot 3 \cdot 5} \right)$$

Outline

$$f^{\lambda} = \frac{|\lambda|!}{\prod_{u \in \lambda} h(u)}$$

$$s_{\lambda}(1, q, \ldots) = \frac{q^{b(\lambda)}}{\prod_{u \in \lambda} (1 - q^{h(u)})}$$

Naruse's formula for $f^{\lambda/\mu}$

$s_{\lambda/\mu}(1, q, \ldots) = ?$

q-analogue Naruse's formula

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- $(*)$ implies Naruse's formula

q-analogue Naruse's formula

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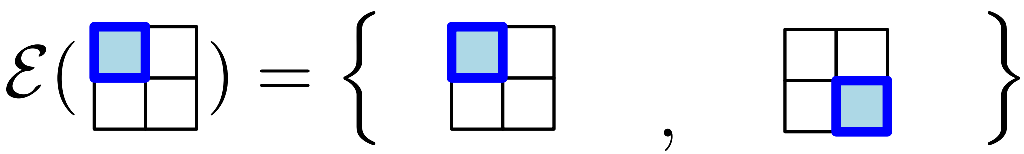
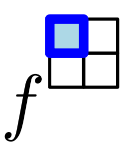
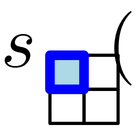


3	2
2	1



$$= 3! \cdot \left(\frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right)$$



$$(1, q, \ldots) = \frac{q^1}{(1-q^1)(1-q^2)^2} + \frac{q^2}{(1-q^2)^2(1-q^3)}$$

Outline

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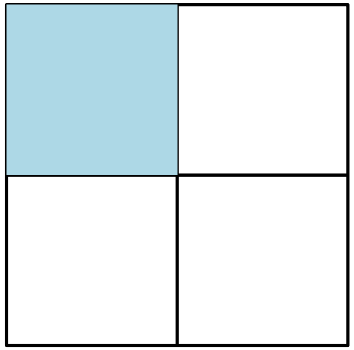
- about algebraic proof,
- about combinatorial proof,
- some questions.

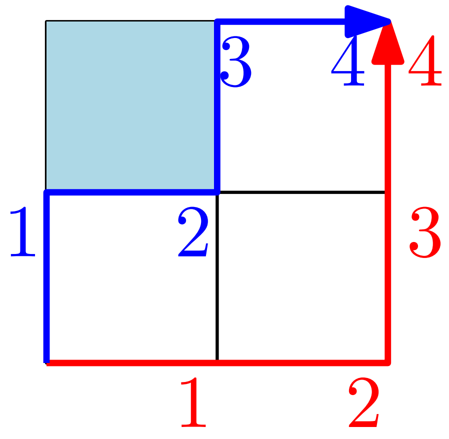
Algebraic proof 1: Grassmannian permutations

$\mu \subseteq \lambda \subseteq d \times (n-d)$ ⟷ $w \leq v$



Algebraic proof 1: Grassmannian permutations

$\mu \subseteq \lambda \subseteq d \times (n-d) \quad \longleftrightarrow \quad w \leq v$



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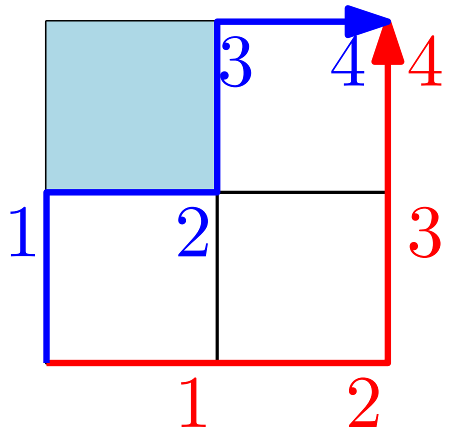

$\longleftrightarrow$

$v = 3412$

$w = 1324$

Algebraic proof 2.

Theorem (Ikeda-Naruse 2009)

$$[X_w]\,|_v = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} \left(\epsilon_{v(d+j)} - \epsilon_{v(d-i+1)}\right),$$

Algebraic proof 2.

Theorem (Ikeda-Naruse 2009)

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where,

$[X_w]$ is *equivariant Schubert class* of w,

$[X_w]\,|_v$ image of $[X_w]$ under certain homomorphism i_v^*,

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- proof uses Billey's formula to write $[X_w]\mid_v$ in terms of **reduced subwords**,

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- proof uses Billey's formula to write $[X_w]\mid_v$ in terms of **reduced subwords**, and Stembridge's theory of **fully commutative elements**.

Algebraic proof 3: factorial Schur functions

The **factorial Schur function** of μ, $\ell(\mu) \leq d$ is

$$s_\mu^{(d)}(x_1, \ldots, x_d \mid a_1, \ldots) := \frac{\det[(x_j - a_1) \cdots (x_j - a_{\mu_i + d - i})]_{i,j=1}^d}{\prod_{1 \leq i < j \leq d} (x_i - x_j)}$$

Algebraic proof 4.

Theorem (Knutson-Tao 03, Lakshmibai-Raghavan-Sankaran 05)

$$[X_w]\,|_v = (-1)^{\ell(w)} \cdot s_\mu^{(d)}(\epsilon_{v(1)}, \ldots, \epsilon_{v(d)} \mid \epsilon_1, \ldots, \epsilon_{n-1}).$$

Algebraic proof: putting everything together

Corollary

$$s_{\mu}^{(d)}(y_{v(1)}, \ldots, y_{v(d)} \mid y_1, \ldots, y_{n-1}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (y_{v(d-i+1)} - y_{v(d+j)}).$$

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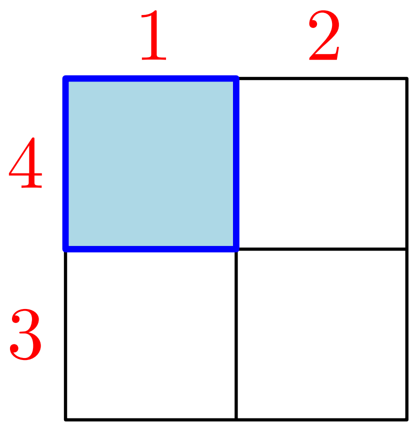
Example



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Algebraic proof: putting everything together

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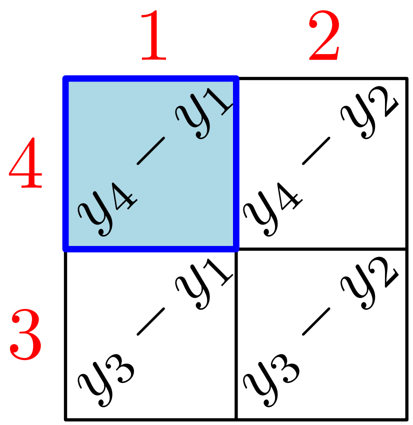

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Algebraic proof: putting everything together

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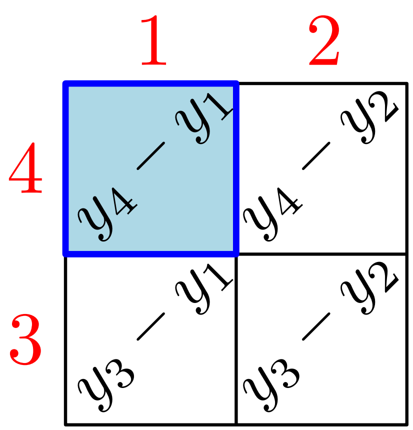
$$s^{(d)}_{\mu}(y_{v(1)},\dots,y_{v(d)} \mid y_1,\dots,y_{n-1}) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (y_{v(d-i+1)} - y_{v(d+j)}).$$

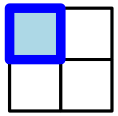
Example

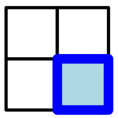


$v = 3412$

$w = 1324$

$$s^{(2)}_{\square}(y_3,y_4 \mid y_1,y_2,y_3,y_4) = (y_4 - y_1) + (y_3 - y_2).$$





Algebraic proof: putting everything together

Corollary

$$s_{\mu}^{(d)}(y_{v(1)}, \ldots, y_{v(d)} \mid y_1, \ldots, y_{n-1}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \left(y_{v(d-i+1)} - y_{v(d+j)}\right).$$

- evaluate $y_p = q^{p-1}$,

Algebraic proof: putting everything together

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- use $\det$ definiton of $s_{\mu}^{(d)}(\cdot \mid \cdot)$,

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Algebraic proof: putting everything together

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Outline

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Naruse's formula for $f^{\lambda/\mu}$

$$s_{\lambda/\mu}(1,q,\ldots) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \ldots$$

- about algebraic proof,
- about combinatorial proof,

From SSYT to reverse plane partitions

Theorem (Stanley 1971)

$$s_\lambda(1, q, q^2, \dots) = q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1 - q^{h(i,j)}}, \qquad (*)$$

From SSYT to reverse plane partitions

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$$\underset{\text{SSYT}}{\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 1 & 5 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array}} \; - \; \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array} \; = \; \underset{\text{RPP}}{\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 & 4 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array}}$$

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equivalently:

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} q^{|\pi|} = \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h(i,j)}}, \qquad (**)$$

$$\underset{\mathrm{SSYT}}{\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 1 & 5 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array}} - \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & \\ \cline{1-2} 2 & & \\ \cline{1-1}\end{array} = \underset{\mathrm{RPP}}{\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 & 4 & \\ \cline{1-2} 1 & & \\ \cline{1-1}\end{array}}$$

Not equivalent for skew shapes:

$$\begin{array}{|c|c|}\hline & 1 \\ \hline 0 & 2 \\ \hline\end{array} - \begin{array}{|c|c|}\hline & 0 \\ \hline 1 & 1 \\ \hline\end{array} = \begin{array}{|c|c|}\hline & 1 \\ \hline -1 & 1 \\ \hline\end{array}$$

Hillman-Grassl bijection

Theorem (Stanley 1971)

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Theorem (Hillman-Grassl 1976)

There is an explicit bijection

$$\mathrm{HG} : \mathrm{RPP}(\lambda) \to \mathcal{A}(\lambda),$$

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- If $\pi \mapsto A$ then $|\pi| = \sum_{u\in\lambda} h(u) \cdot A_u$.

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Example

π

0	1
2	2

A

0	0
0	0

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π			A	
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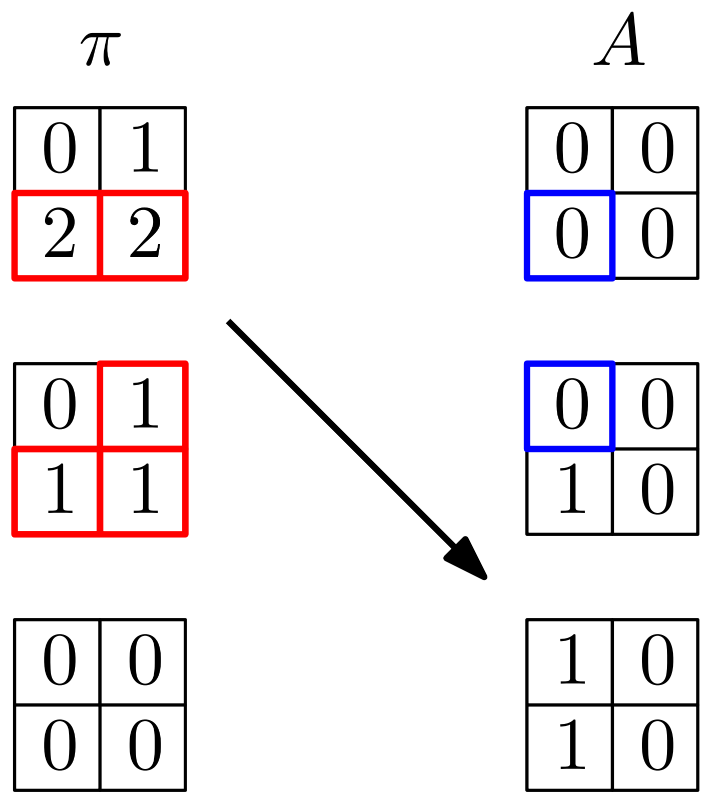

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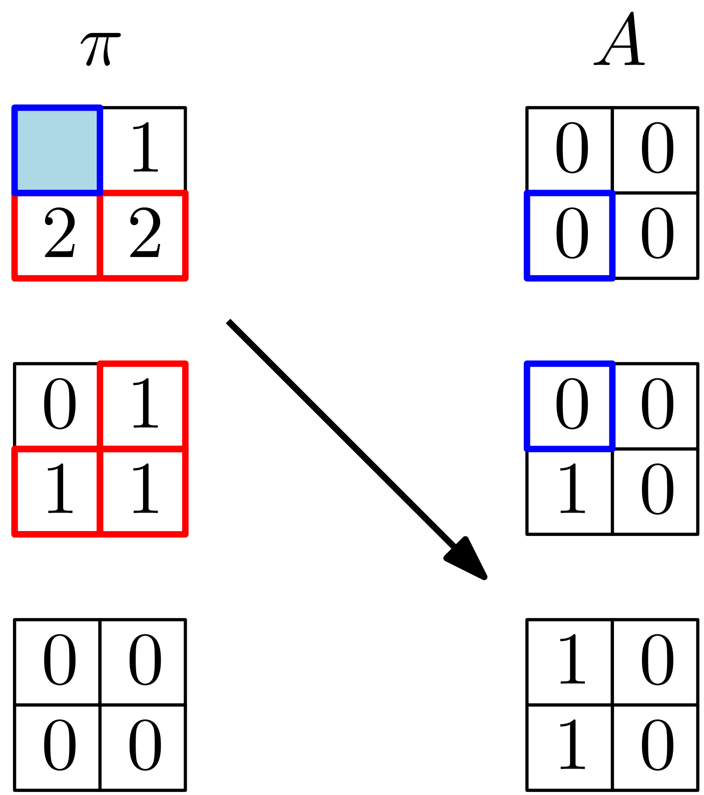

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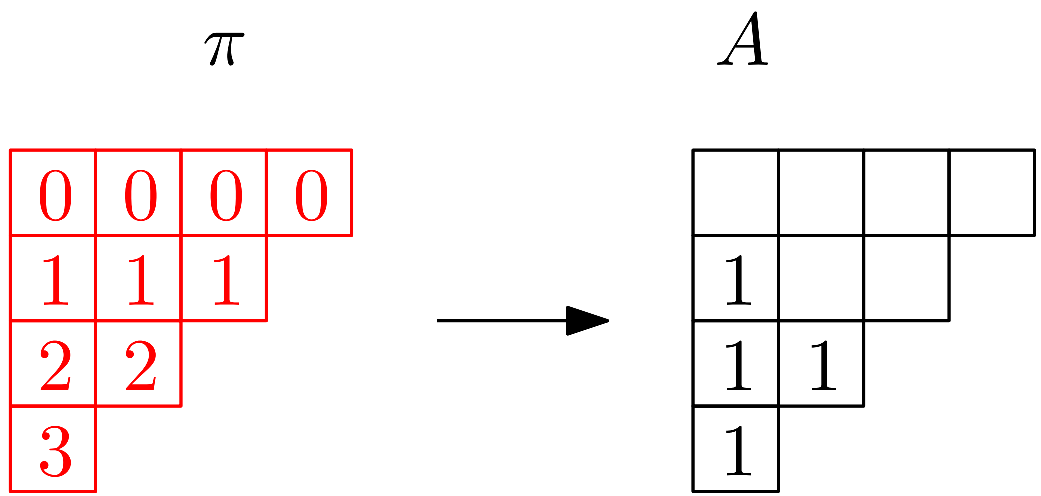

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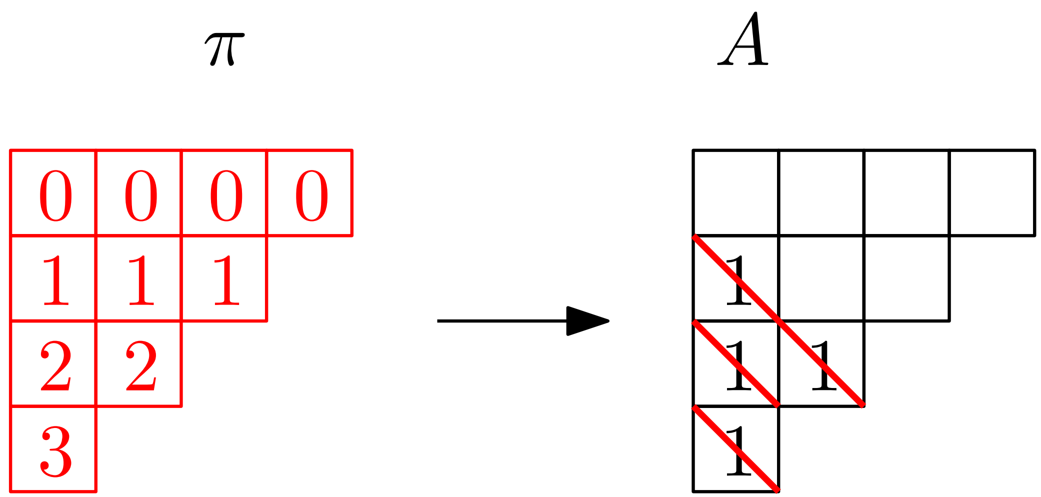

Hillman-Grassl on $\mathrm{SSYT}(\lambda/\mu)$

Theorem (M-Pak-Panova 2015+)

The HG map is a bijection:

$$\mathrm{HG} : \mathrm{SSYT}(\lambda/\mu) \to \bigcup_{D \in \mathcal{E}(\lambda/\mu)} \mathcal{A}^*(\lambda \setminus D),$$

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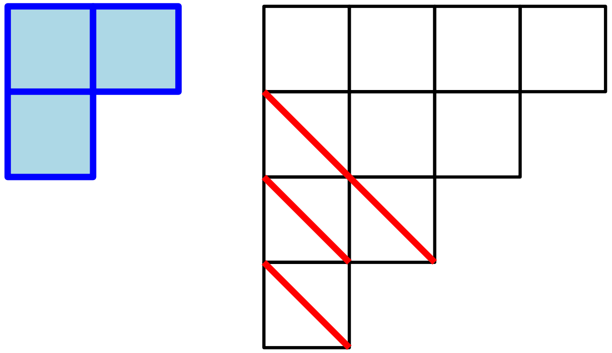

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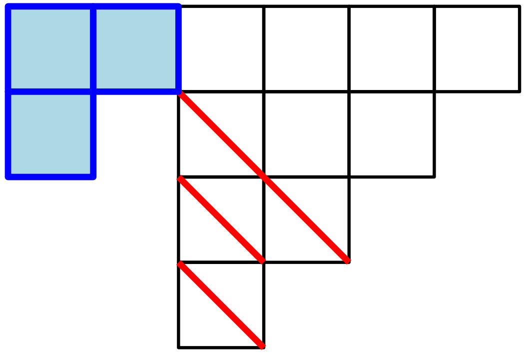

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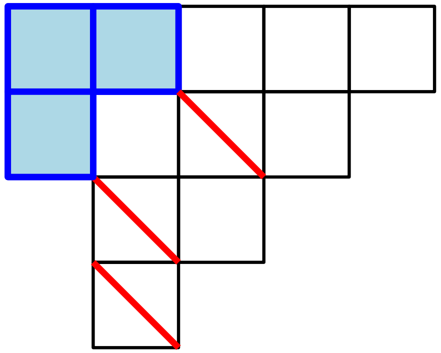

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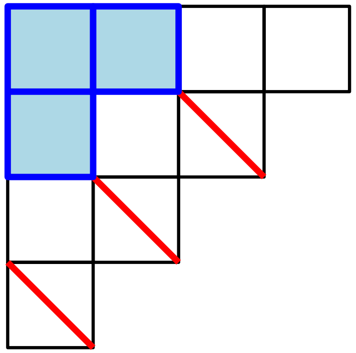

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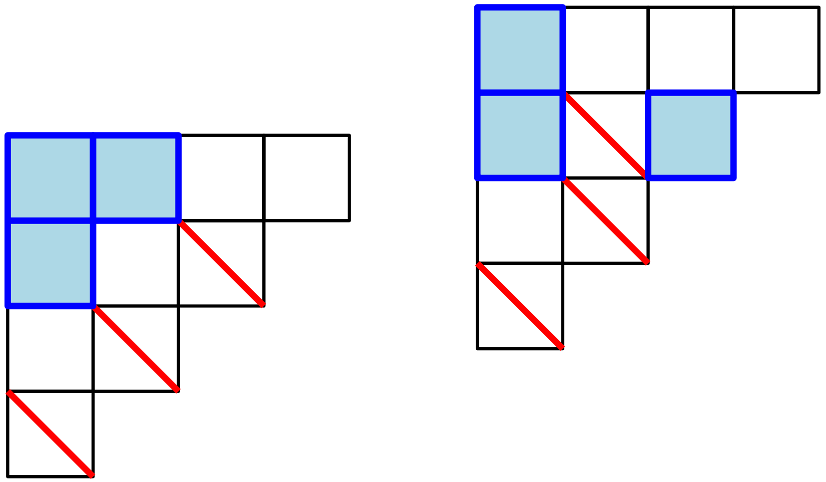

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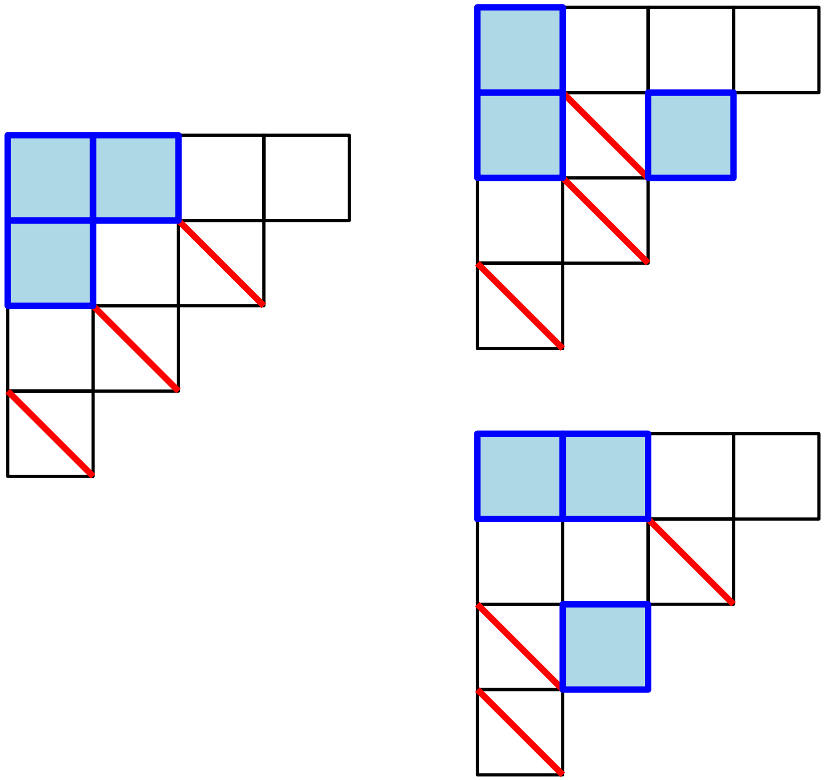

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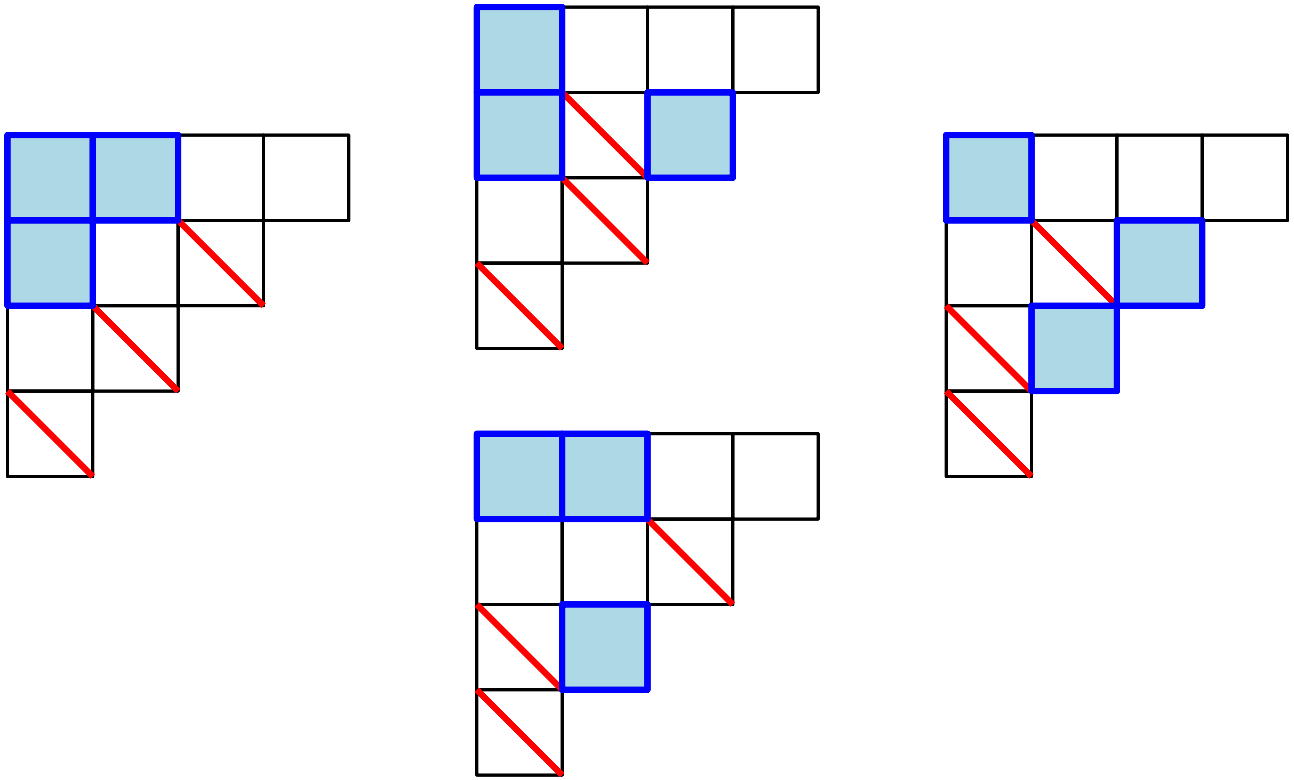

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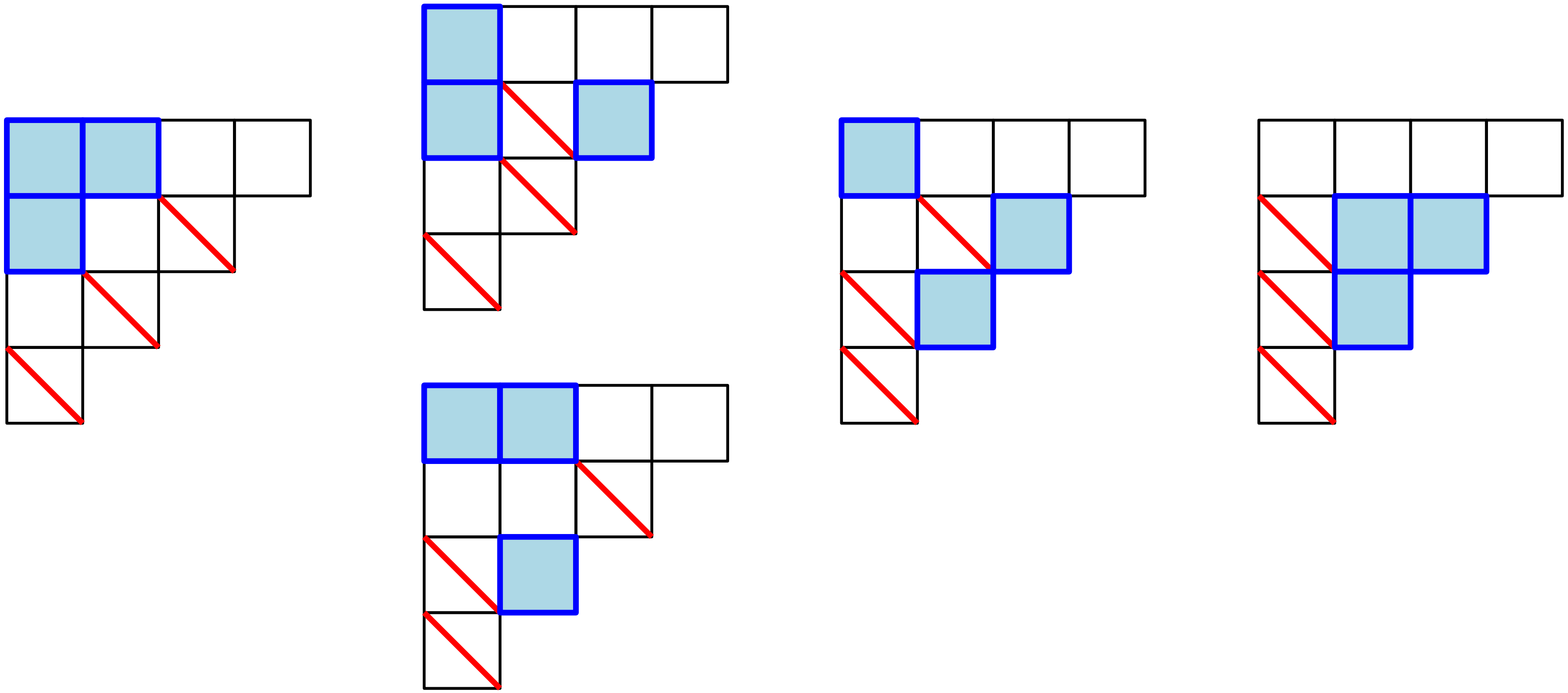

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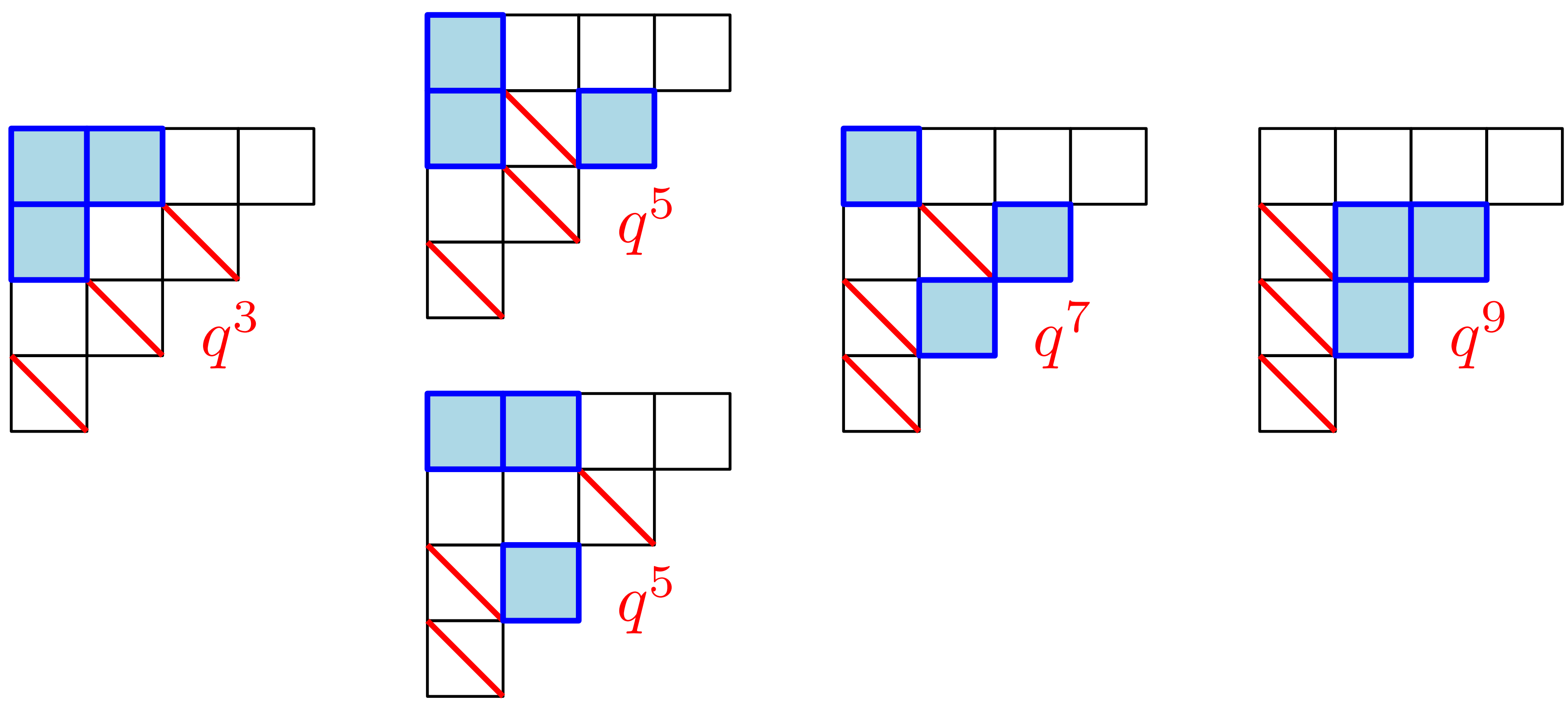

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Recall:

$$s_{\lambda/\mu}(1, q, , q^2, \ldots) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \Bigg(\prod_{(i,j) \in \lambda \setminus D} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \Bigg).$$

Hillman-Grassl on $\mathrm{RPP}(\lambda/\mu)$

Theorem (M-Pak-Panova 2015+)

The HG map is a bijection:

$$\mathrm{HG} : \mathrm{RPP}(\lambda/\mu) \to \bigcup_{D \in \mathcal{P}(\lambda/\mu)} \mathcal{A}^{**}(D),$$

$\mathcal{A}^{**}(D) := \{\mathbb{N}\text{-arrays, nonzero eactly in } D\}$,
$\mathcal{P}(\lambda/\mu)$ set of **pleasant diagrams** of λ/μ.

Outline

$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

$$s_\lambda(1,q,\ldots) = \frac{q^{b(\lambda)}}{\prod_{u\in\lambda}(1-q^{h(u)})}$$

Naruse's formula for $f^{\lambda/\mu}$

$$s_{\lambda/\mu}(1,q,\ldots) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \cdots$$

- about algebraic proof,
- about combinatorial proof,
- some questions.

I. Hook-content formula?

Theorem (Stanley 1971)

$$s_\lambda(1^n) = \prod_{(i,j)\in\lambda} \frac{n + c(i,j)}{h(i,j)},$$

where $c(i,j) = j - i$.

I. Hook-content formula?

Theorem (Stanley 1971)

$$s_\lambda(1, q, \ldots, q^{n-1}) = q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{1 - q^{n+c(i,j)}}{1 - q^{h(i,j)}},$$

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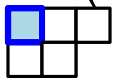
Is there an Naruse type analogue for λ/μ?

Example

s  $(1^n) = \dfrac{n(n-1)(n+1)^2}{1^2 \cdot 2 \cdot 3} + \dfrac{n(n-1)(n+1)(n+2)}{1 \cdot 2 \cdot 3 \cdot 4}$

4	3	1
2	1	

	1	1
−1	0	

0	1	2
−1		

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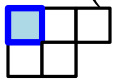
Is there an Naruse type analogue for λ/μ?

Example

$$s \quad (1, \ldots, q^{n-1}) = q^{1} \frac{[n][n-1][n+1]^2}{[1]^2 \cdot [2] \cdot [3]} + q^{2} \frac{[n][n-1][n+1][n+2]}{[1] \cdot [2] \cdot [3] \cdot [4]}$$



where $[m] := 1 - q^m$

4	3	1
2	1	

	1	1
−1	0	

0	1	2
−1		

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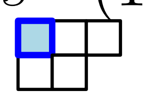
Is there an Naruse type analogue for λ/μ?

What are the (excited) contents?

Example

$$s_{\square}(1, \ldots, q^{n-1}) = q^{1} \frac{[n][n-1][n+1]^2}{[1]^2 \cdot [2] \cdot [3]} + q^{2} \frac{[n][n-1][n+1][n+2]}{[1] \cdot [2] \cdot [3] \cdot [4]}$$



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4	3	1
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II. q-analogue of Naruse's skew shifted formula

Theorem (Naruse 2014)

$\mu \subset \lambda$ strict partitions,

$$g^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}'(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h'(i,j)},$$

where $\mathcal{E}'(\lambda/\mu)$ is the set of **shifted excited diagrams** of λ/μ.

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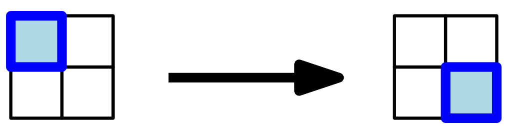
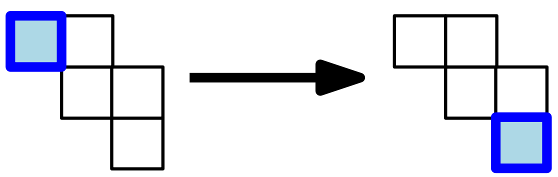
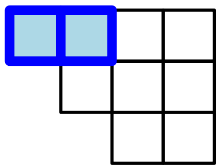




Example

$(4,3,2)/(2)$



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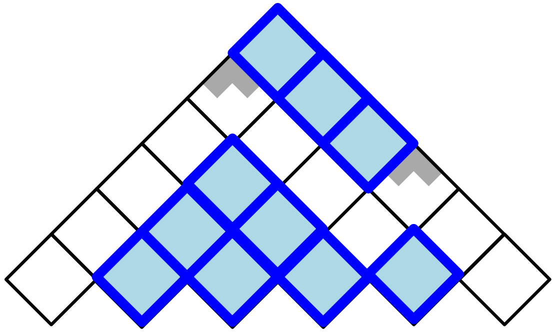
Thank you!



References

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- **Excited Young diagrams and equivariant Schubert Calculus**, T. Ikeda, H. Naruse, Trans. Amer. Math. Soc., 2009
- **q-analogues of Naruse's hook-length formula for skew shapes**, M., I. Pak, G. Panova, coming soon!