Schur positivity arising from log-concavity problems

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joint work with

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Outline

Unimodality

Let $\{a_i\}_{0\le i\le m}$ be a positive sequence of real numbers.

Definition

$\{a_i\}_{0\le i\le m}$ is unimodal if there exists k such that

$$a_0 \le \cdots \le a_k \ge \cdots \ge a_m,$$

and is strictly unimodal if

$$a_0 < \cdots < a_k > \cdots > a_m.$$

Example

For fixed m, $\{\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\}$ is symmetric and unimodal. Furthermore, it is strictly unimodal if m is even.

Log-concavity

Definition

$\{a_i\}_{0 \le i \le m}$ is *log-concave* if

$$a_i^2 \ge a_{i+1} a_{i-1}$$

for all $1 \le i \le m-1$, and is *strictly log-concave* if

$$a_i^2 > a_{i+1} a_{i-1}.$$

$f(x)$

Remark: A log-concave sequence is unimodal.

Example

For fixed m, $\{\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\}$ *is strictly log-concave. While* $\{1, 3, 5, 9, 5, 3, 1\}$ *is unimodal, but not log-concave.*

Let $f(q) = a_0 + a_1 q + \cdots + a_m q^m$ be a polynomial with real coefficients.

Definition

$f(q)$ is unimodal (or strictly unimodal) if $\{a_i\}_{0\le i\le m}$ is unimodal (resp. strictly unimodal).

Definition

$f(q)$ is log-concave (or strictly log-concave) if $\{a_i\}_{0\le i\le m}$ is log-concave (resp. strictly log-concave).

Example

Let $des(\pi)$ denote the number of descents of π. The Eulerian polynomial $A_m(q) = \sum_{\pi\in\mathfrak{S}_m} q^{1+des(\pi)}$ is strictly log-concave.

q-Log-concavity

Let $\{f_i(q)\}_{0\leq i\leq m}$ be a sequence of polynomials with real coefficients.

Definition

For any two polynomials $f(q)$ and $g(q)$ with real coefficients, define $f(q)\geq_q g(q)$ if and only if $f(q)-g(q)$, as a polynomial in q, has all nonnegative coefficients.

Definition

$\{f_i(q)\}_{0\leq i\leq m}$ *is q-log-concave if*

$$f_i(q)^2 \geq_q f_{i+1}(q)f_{i-1}(q), \quad 1\leq i\leq m-1,$$

and is strongly q-log-concave if

$$f_i(q)f_j(q) \geq_q f_{i+1}(q)f_{j-1}(q), \quad i\geq j\geq 1.$$

q-Log-concavity

Example

The Gaussian binomial coefficients $\{\begin{bmatrix} m \\ k \end{bmatrix}_q\}_{0 \le k \le m}$ are strongly q-log-concave.

- *The q-log-concavity was conjectured by Butler (1987).*
- *The first proof was given by Butler (1990).*
- *Krattenthaler (1989) found an alternative combinatorial proof.*
- *Sagan (1992) gave an inductive proof.*

Remark: Usually, a q-log-concave sequence is not strongly q-log-concave.

Example

The sequence $\{q^2, q + q^2, 1 + 2q + q^2, 4 + q + q^2\}$ is q-log concave but not strongly q-log concave.

q-Log-convexity

Based on the q-log-concavity, it is natural to define the q-log-convexity.

Definition

$\{f_i(q)\}_{0\leq i\leq m}$ *is q-log-convex if*

$$f_i(q)^2 \leq_q f_{i+1}(q)f_{i-1}(q), \quad 1 \leq i \leq m-1,$$

and is strongly q-log-convex if

$$f_i(q)f_j(q) \leq_q f_{i+1}(q)f_{j-1}(q), \quad i \geq j \geq 1.$$

Example

The sequence
$\{2q+q^2+3q^3, q+2q^2+2q^3, q+2q^2+2q^3, 2q+q^2+3q^3\}$ *is q-log-convex, but not strongly q-log-convex.*

Young Diagram

Let λ be a partition of n.

Definition

The Young diagram of λ is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and λ_i squares in row i.

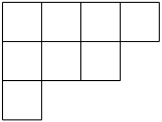


Fig 1: The diagram of $(4, 3, 1)$

Semistandard Young Tableau

Definition

A semistandard Young tableau (SSYT) of shape λ/μ is an array $T = (T_{ij})$ of positive integers of shape λ/μ that is weakly increasing in every row and strictly increasing in every column.

The type of T is defined as the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T.

1	1	2	3
2	2	4	
3			

Fig 2: SSYT of shape $(4, 3, 1)$

Schur Function

If T has type $\mathrm{type}(T) = \alpha$, then we write

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

Definition

The Schur function $s_\lambda(x)$ of shape λ is defined as the generating function

$$s_\lambda(x) = \sum_T x^T,$$

summed over all semistandard Young tableaux T of shape λ. We set $s_\emptyset(x) = 1$.

Schur Positivity

Theorem

The Schur functions $s_\lambda(x)$ are symmetric functions, and $\{s_\lambda(x) \mid \lambda \vdash n\}$ form a basis of symmetric functions of degree n.

Definition

Given a symmetric function $f(x)$, we say that it is Schur positive if all the coefficients are positive when expanding $f(x)$ in terms of Schur functions.

For a symmetric function $f(x)$, define

$$\begin{aligned} \mathrm{ps}_n(f) &= f(1, q, \ldots, q^{n-1}), \\ \mathrm{ps}_n^1(f) &= \mathrm{ps}_n(f)|_{q=1} = f(1^n). \end{aligned}$$

Outline

q-Narayana Numbers

The q-Narayana numbers, as a natural q-analogue of the Narayana numbers $N(n,k)$, arise in the study of q-Catalan numbers. The q-Narayana number $N_q(n,k)$ is given by

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{k^2-k},$$

where we have adopted the common notation

$$[k] := (1-q^k)/(1-q), \quad [k]! = [1][2]\cdots[k], \quad \begin{bmatrix} n \\ j \end{bmatrix} := \frac{[n]!}{[j]![n-j]!}$$

for the q-analogues of the integer k, the q-factorial, and the q-binomial coefficient, respectively.

Hook-content Formula

A square (i, j) in λ is the square in row i from the top and column j from the left. The hook length $h(i, j)$, is given by $\lambda_i + \lambda'_j - i - j + 1$. The content $c(i, j)$ is given by $j - i$.

Theorem (Stanley, Studies in Applied Math. (1971))

For any partition λ and $n \geq 1$, we have

$$\begin{aligned} \mathrm{ps}_n(s_\lambda) &= q^{\sum_{k\geq 1}(k-1)\lambda_k} \prod_{(i,j)\in\lambda} \frac{[n + c(i,j)]}{[h(i,j)]} \\ \mathrm{ps}_n^1(s_\lambda) &= \prod_{(i,j)\in\lambda} \frac{n + c(i,j)}{h(i,j)}. \end{aligned}$$

Brändén's Formula for q-Narayana Numbers

Brändén noticed that the q-Narayana number $N_q(n,k)$ has a Schur function expression by a specialization of the variables.

Theorem (Brändén, Discrete Math. (2004))

For all $n, k \in \mathbb{N}$, we have

$$N_q(n,k) = s_{(2^{k-1})}(q, q^2, \ldots, q^{n-1}).$$

Thus

$$N(n,k) = N_q(n,k)|_{q=1} = s_{(2^k)}(1^{n-1}) = \mathrm{ps}^1_{n-1} s_{(2^k)}.$$

q-Log-concavity of $N_q(n,k)$ for Fixed n

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

Given an integer n, the sequence $(N_q(n,k))_{k\geq 1}$ of polynomials in q is strongly q-log-concave.

For any $k \geq l \geq 2$,

$$N_q(n,k)N_q(n,l) - N_q(n,k+1)N_q(n,l-1) = s_{(2^{k-1})}s_{(2^{l-1})} - s_{(2^k)}s_{(2^{l-2})},$$

where the Schur functions are evaluated at the variable set $\{q, q^2, \ldots, q^{n-1}\}$.

Theorem (Bergeron-McNamara, 2004, arXiv)

For $k \geq 1$ and $a \geq b$, the symmetric function $s_{(k^a)}s_{(k^b)} - s_{(k^{a+1})}s_{(k^{b-1})}$ is Schur positive.

The case of $a = b$ is due to Kirillov (1984), and a different proof was given by Kleber (2001).

q-Log-concavity of $N_q(n,k)$ for Fixed k

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

Given an integer k, the sequence $(N_q(n,k))_{n\geq k}$ is strongly q-log-concave.

Proof. For any $m \geq n \geq k$, the difference $N_q(m,k)N_q(n,k) - N_q(m+1,k)N_q(n-1,k)$ equals

$$q^{k-2}s_{(2^{k-2},1)}(X_{n-1})s_{(2^{k-1})}(Z) + q^{2(k-1)(m+n-1)}s_{(2^{k-2})}(X_{n-1}^{-1})s_{(2^{k-1})}(Z^{-1})$$

$$+ q^{k-2}\sum_{J\subseteq(2^{k-2},1)} s_J(Z)\left(s_{(2^{k-2},1)}s_{(2^{k-1})/J} - s_{(2^{k-2},1)/J}s_{(2^{k-1})}\right)(X_{n-1})$$

$$+ q^{2(k-1)(m+n-1)}s_{(2^{k-2})}(X_{n-1}^{-1})s_{(2^{k-2},1)}(Z^{-1})s_{(1)}(X_{n-1}^{-1})$$

$$+ q^{2(k-1)(m+n-1)}\sum_{I\subseteq(2^{k-2})} s_I(Z^{-1})\left(s_{(2^{k-2})}s_{(2^{k-1})/I} - s_{(2^{k-2})/I}s_{(2^{k-1})}\right)(X_{n-1}^{-1}),$$

where $X_r = \{q, q^2, \ldots, q^{r-1}\}$, $X_r^{-1} = \{q^{-1}, q^{-2}, \ldots, q^{-(r-1)}\}$, $Z = \{q^{n-1}, \ldots, q^{m-1}\}$ and $Z^{-1} = \{q^{1-n}, \ldots, q^{1-m}\}$.

q-Log-concavity of $N_q(n,k)$ for Fixed k

Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, let

$$\begin{aligned} \lambda \vee \mu &= (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \ldots), \\ \lambda \wedge \mu &= (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots). \end{aligned}$$

For two skew partitions λ/μ and ν/ρ, we define

$$\begin{aligned} (\lambda/\mu) \vee (\nu/\rho) &= (\lambda \vee \nu)/(\mu \vee \rho), \\ (\lambda/\mu) \wedge (\nu/\rho) &= (\lambda \wedge \nu)/(\mu \wedge \rho). \end{aligned}$$

Theorem (Lam-Postnikov-Pylyavaskyy, Amer. J. Math. (2007))

For any two skew partitions λ/μ and ν/ρ, the difference

$$s_{(\lambda/\mu)\vee(\nu/\rho)} s_{(\lambda/\mu)\wedge(\nu/\rho)} - s_{\lambda/\mu} s_{\nu/\rho}$$

is Schur positive.

q-Log-concavity of $N_q(n,k)$ for Fixed k

Corollary

Let k be a positive integer. If I, J are partitions with $I \subseteq (2^{k-1})$ and $J \subseteq (2^{k-1}, 1)$, then both

$$s_{(2^{k-1})} s_{(2^k)/I} - s_{(2^{k-1})/I} s_{(2^k)} \tag{1}$$

and

$$s_{(2^{k-1},1)} s_{(2^k)/J} - s_{(2^{k-1},1)/J} s_{(2^k)} \tag{2}$$

are Schur positive.

Proof. For (1), take $\lambda = (2^{k-1}), \mu = I, \nu = (2^k)$ and $\rho = \emptyset$. For (2), take $\lambda = (2^{k-1}, 1), \mu = J, \nu = (2^k)$ and $\rho = \emptyset$.
Remark. The q-Log-Concavity of $N_q(n,k)$ for fixed k follows from the above corollary.

Connection with a Conjecture of McNamara and Sagan

Define the operator $\mathcal{L}$ which maps a polynomial sequence $\{f_i(q)\}_{i\geq 0}$ to a polynomial sequence given by

$$\mathcal{L}(f_i(q)) := f_i(q)^2 - f_{i-1}(q)f_{i+1}(q).$$

A sequence $\{f_i(q)\}$ is k-fold q-log-concave if $\mathfrak{L}^j(f_i)$ is q-log-concave for $1 \leq j \leq k-1$.
If $\{f_i(q)\}$ is k-fold log-concave for any k, then it is said to be infinitely q-log-concave.

Conjecture (McNamara and Sagan, Adv. in Appl. Math. (2010))

For fixed k, the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_{n\geq k}$ is infinitely q-log-concave.

Remark. For fixed n, they have shown that $\begin{bmatrix} n \\ k \end{bmatrix}_k$ is not 2-fold q-log-concave.

Connection with a Conjecture of McNamara and Sagan

For fixed k, subscript the $\mathcal{L}$-operator by n.

$$\mathfrak{L}_n\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{n-k}}{[n]}\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix},$$

which are, up to a power of q, the q-Narayana numbers.

$$\mathfrak{L}_n^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{3n-3k}[2]}{[n]^2[n-1]}\begin{bmatrix} n \\ k \end{bmatrix}^2\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ k-2 \end{bmatrix}.$$

McNamara and Sagan (2010) conjectured that these polynomials are q-nonnegative.

Connection with a Conjecture of McNamara and Sagan

McNamara and Sagan (2010):
"It is not clear that these polynomials are q-nnonegative, although they must be if Conjecture 5.3 is true. Furthermore, when $q = 1$, the triangle made as n and k vary is not in Sloane's Encyclopedia [24] (although it has now been submitted). We expect that these integers and polynomials have interesting, yet to be discovered, properties."

Corollary (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For fixed k, the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_{n \geq k}$ *is 2-fold q-log-concave.*

Further Result

Let $X_n = \{q,\, q^2,\, \ldots,\, q^{n-1}\}$.

Theorem (King-Yang, preprint)

For any partition λ, the polynomial sequence $\{s_\lambda(X_n)\}_{n\geq 1}$ is strongly q-log-convex. Namely, for any $n \geq m \geq 1$, we have

$$s_\lambda(X_m)\; s_\lambda(X_n) - s_\lambda(X_{m-1})\; s_\lambda(X_{n+1}) \geq_q 0\,.$$

Proof.

$$s_\lambda(X_m)\; s_\lambda(X_n) - s_\lambda(X_{m-1})\; s_\lambda(X_{n+1})$$

$$= \sum_{\rho,\mu:\lambda/\mu=h.s} q^A\, s_\rho(X_n/X_{m-1})\; \left(s_\mu(X_{m-1}^{-1})\; s_{\lambda/\rho}(X_{m-1}^{-1}) - s_\lambda(X_{m-1}^{-1})\; s_{\mu/\rho}(X_{m-1}^{-1})\right.$$

where

$$A = (n+m-1)|\lambda| + f(\mu) + f(\lambda).$$

Schur Positivity

Recall that

$$s_{(\lambda\wedge\nu)/(\mu\wedge\rho)}\ s_{(\lambda\vee\nu)/(\mu\vee\rho)} - s_{\lambda/\mu}\ s_{\nu/\rho} \geq_s 0\,.$$

If $\nu \subseteq \lambda$ and $\mu \subseteq \rho$ then

$$s_{\lambda/\rho}\ s_{\nu/\mu} - s_{\lambda/\mu}\ s_{\nu/\rho} \geq_s 0.$$

Setting $\mu = 0$ and then $\nu = \mu$, we find

$$s_{\lambda/\rho} s_{\mu} - s_{\lambda}\ s_{\mu/\rho} \geq_s 0$$

for any $\mu \subseteq \lambda$.

Outline

q-Log-convexity of Narayana Polynonmials

Narayana polynomial of type A and B are defined respectively as follows:

$$NA_n(q) = \sum_{k=0}^{n} N(n,k)q^k,$$

and

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$

Conjecture (Liu-Wang, Adv. in Appl. Math. (2007))

The polynomials $NA_n(q)$ form a q-log-convex sequence, so do $NB_n(q)$.

q-Log-convexity of Narayana Polynonmials

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

The Narayana polynomials $NA_n(q)$ of type A are strongly q-log-convex.

Theorem (Chen-Tang-Wang-Yang, Adv. in Appl. Math. (2010))

The Narayana polynomials $NB_n(q)$ of type B are q-log-convex.

Idea: q-log-convexity $\Rightarrow$ Schur positivity
Method: regard coefficients as specialization of symmetric functions.
Remark: Zhu (Adv. in Appl. Math., 2013) gave a simple proof of the q-log-convexity of Narayana polynomials by using the recurrence relations.

Narayana Polynonmials of Type A

$$N(n,k) = N_q(n,k)|_{q=1} = s_{(2^{k-1})}(1^{n-1}) = \mathrm{ps}^1_{n-1}\left(s_{(2^{k-1})}\right).$$

$$[q^r] NA_{m+1}(q) NA_{n-1}(q) = \sum_{k=0}^{r-2} \mathrm{ps}^1_m\left(s_{(2^k)}\right)\mathrm{ps}^1_{n-2}\left(s_{(2^{r-2-k})}\right).$$

$$[q^r] NA_m(q) NA_n(q) = \sum_{k=0}^{r-2} \mathrm{ps}^1_{m-1}\left(s_{(2^k)}\right)\mathrm{ps}^1_{n-1}\left(s_{(2^{r-2-k})}\right)$$

Narayana Polynonmials of Type A

Given $a, b, m \in \mathbb{N}$ and $0 \le i \le m$, let

$$\begin{aligned} D_1(m,i,a,b) &= s_{(2^{i-b},1^{b-a})} s_{(2^{m-i-1})}, \\ D_2(m,i,a,b) &= s_{(2^{i-b-1},1^{b+2-a})} s_{(2^{m-i-1})}, \\ D_3(m,i,a,b) &= s_{(2^{i-b-1},1^{b+1-a})} s_{(2^{m-i-1},1)}, \end{aligned}$$

$$D(m,i,a,b) = D_1(m,i,a,b) + D_2(m,i,a,b) - D_3(m,i,a,b).$$

The coefficient $[q^r]\,(NA_{m+1}(q)NA_{n-1}(q) - NA_m(q)NA_n(q))$ is equal to

$$\mathrm{ps}^1_{n-2}\left(\sum_{0\le a\le b\le d-1} \mathrm{ps}^1_d(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r-2} D(r-2,k,a,b) \right).$$

Schur Positivity

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For any $b \geq a \geq 0$ and $m \geq 0$, the symmetric function $\sum_{i=0}^{m} D(m, i, a, b)$ is Schur positive.

Proof is based on the case of $a = b = 0$.

Given a set S of positive integers, let $\mathrm{Par}_S(n)$ denote the set of partitions of n whose parts belong to S.

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For any $m \geq 0$, we have

$$\sum_{i=0}^{m} D(m, i, 0, 0) = \sum_{\lambda \in \mathrm{Par}_{\{2,4\}}(2m-2)} s_\lambda. \tag{3}$$

Schur Positivity

Taking $m = 3, 4, 5$ and using the Maple package, we observe that

$$\sum_{k=0}^{3} \left(s_{(2^{k-1})}s_{(2^{3-k})} + s_{(2^{k-2},1^2)}s_{(2^{3-k})} - s_{(2^{k-1},1)}s_{(2^{3-k-1},1)}\right) = s_{(4)} + s_{(2,2)}.$$

$$\sum_{k=0}^{4} \left(s_{(2^{k-1})}s_{(2^{4-k})} + s_{(2^{k-2},1^2)}s_{(2^{4-k})} - s_{(2^{k-1},1)}s_{(2^{4-k-1},1)}\right) = s_{(4,2)} + s_{(2,2,2)}.$$

$$\sum_{k=0}^{5} \left(s_{(2^{k-1})}s_{(2^{5-k})} + s_{(2^{k-2},1^2)}s_{(2^{5-k})} - s_{(2^{k-1},1)}s_{(2^{5-k-1},1)}\right) = s_{(4,4)} + s_{(4,2,2)} + s_{(2,2,2,2)}.$$

The proof of the above theorem mainly relies on the recurrence relations of summands $D(m, i, 0, 0)$.

Experiment $\Rightarrow$ Observation $\Rightarrow$ Proof

Narayana Polynomials of Type B

When $\lambda = (1^k)$ for $k \geq 1$, the Schur function $s_\lambda(x)$ becomes the k-th elementary symmetric function $e_k(x)$, i.e.,

$$s_{(1^k)}(x) = e_k(x) = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \quad (4)$$

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$

$$[q^k](NB_n(q)) = \mathrm{ps}_n^1(e_k^2).$$

$$\mathrm{ps}_n^1(e_k) = \mathrm{ps}_{n-1}^1(e_k + e_{k-1}).$$

Narayana Polynomials of Type B

The coefficient of q^r in $NB_{n-1}(q)NB_{n+1}(q) - (NB_n(q))^2$ is given by

$$\sum_{k=0}^{r} \mathrm{ps}_{n-1}^{1}(e_k)^2 \mathrm{ps}_{n+1}^{1}(e_{r-k})^2 - \mathrm{ps}_{n}^{1}(e_k)^2 \mathrm{ps}_{n}^{1}(e_{r-k})^2.$$

$\Downarrow$ apply $\mathrm{ps}_n^1(e_k) = \mathrm{ps}_{n-1}^1(e_k + e_{k-1})$ twice.

$$\mathrm{ps}_{n-1}^{1}\left(\sum_{k=0}^{r} {e_k}^2(e_{r-k} + 2e_{r-k-1} + e_{r-k-2})^2 - (e_k + e_{k-1})^2(e_{r-k} + e_{r-k-1})^2\right).$$

$$\Downarrow$$

$$2\,\mathrm{ps}_{n-1}^{1}\left(\sum_{k=0}^{r} e_{k-1}^2 e_{r-k}^2 + e_{k-2}e_k e_{r-k}^2 - 2e_{k-1}e_k e_{r-k-1}e_{r-k}\right).$$

Narayana Polynomials of Type B

Theorem (Chen-Tang-Wang-Yang, Adv. in Appl. Math. (2010))

For any $r \geq 1$, we have

$$\sum_{k=0}^{r}(e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_k e_{r-k}e_{r-k} - 2e_{k-1}e_k e_{r-k-1}e_{r-k}) = \sum_{\lambda} s_\lambda,$$

where λ sums over all partitions of $2r-2$ of the form $(4^{i_4}, 3^{2i_3}, 2^{2i_2}, 1^{2i_1})$ with i_1, i_2, i_3, i_4 being nonnegative integers.

Remark. Proof relies on the Jacobi-Trudi identity.

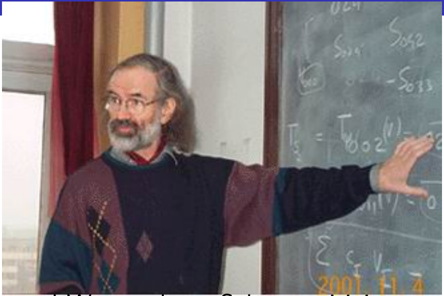
Theorem (The Jacobi-Trudi identity)

Let λ be a partition with the largest part $\leq n$ and λ' its conjugate. Then

$$s_\lambda(x) = \det(e_{\lambda_i'-i+j}(x))_{i,j=1}^n,$$

where $e_0 = 1$ and $e_k = 0$ for $k < 0$.

Simple proofs?



From: Alain Lascoux⟨ al@univ-mlv.fr⟩
To: Arthur⟨ yang@nankai.edu.cn⟩
Cc:
Subject: Schur Positivity
Date: Sun, 6 Jul 2008 18:27:44

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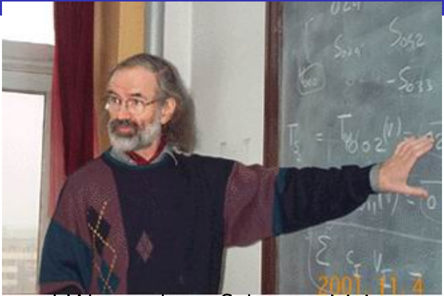
Simple proofs?



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A quick look at your article reminds me that there are many things that I did not finish, in particular in my course, the use of symmetrizing operators in symmetric function theory. As an example, I shall take your functions $D(m, r)$ p.9. It is more convenient to transpose partitions. ......

Outline

Longest Increasing Subsequences

Let

$$P_n(q) = \sum_k P_{n,k} q^k,$$

where $P_{n,k}$ is the number of permutations π on $[n] = \{1, 2, \ldots, n\}$ such that the length of the longest increasing subsequences of π equals k.

Theorem (Baik-Deift-Johansson, J. Amer. Math. Soc. (1999))

The limiting distribution of the coefficients of $P_n(q)$ is the Tracy-Widom distribution.

The numbers $P_{n,k}$ can be computed by Gessel's theorem. Let $\mathfrak{S}_n$ be the symmetric group on $[n]$, and let $\mathrm{is}(\pi)$ be the length of the longest increasing subsequences of π.

Longest Increasing Subsequences

Define

$$u_k(n) = \sharp\{w \in \mathfrak{S}_n : \mathrm{is}(w) \leq k\}, \tag{5}$$

$$U_k(q) = \sum_{n\geq 0} u_k(n)\frac{q^{2n}}{n!^2}, \quad k \geq 1, \tag{6}$$

$$I_i(2q) = \sum_{n\geq 0} \frac{q^{2n+i}}{n!(n+i)!}, \quad i \in \mathbb{Z}. \tag{7}$$

Theorem (Gessel, J. Combin. Theory, Ser. A (1990))

$$U_k(q) = \det(I_{i-j}(2q))_{i,j=1}^{k}.$$

Longest Increasing Subsequences

Note that $P_{n,k} = u_k(n) - u_{k-1}(n)$ for $n \geq 1$.

$$P_1(q) = q,$$

$$P_2(q) = q + q^2,$$

$$P_3(q) = q + 4q^2 + q^3,$$

$$P_4(q) = q + 13q^2 + 9q^3 + q^4,$$

$$P_5(q) = q + 41q^2 + 61q^3 + 16q^4 + q^5,$$

$$P_6(q) = q + 131q^2 + 381q^3 + 181q^4 + 25q^5 + q^6,$$

$$P_7(q) = q + 428q^2 + 2332q^3 + 1821q^4 + 421q^5 + 36q^6 + q^7.$$

Longest Increasing Subsequences

Conjecture

$P_n(q)$ is log-concave for $n \geq 1$.

Conjecture

$P_n(q)$ is ∞-log-concave for $n \geq 1$.

Conjecture

The polynomial sequence $\{P_n(q)\}$ is strongly q-log-convex.

Conjecture

The polynomial sequence $\{P_n(q)\}$ is infinitely q-log-convex.

These conjectures were proposed by W.Y.C. Chen (unpublished).

Longest Increasing Subsequences

Let $f^{\lambda/\mu}$ denote the number of standard Young tableaux of shape λ/μ. The exponential specialization is a homomorphism $ex : \Lambda \to \mathbb{Q}[t]$, defined by $ex(p_n) = t\delta_{1n}$, where p_n is the n-th power sum. Let $ex_1(f) = ex(f)_{t=1}$, provided this number is defined. It is known that

$$ex_1(s_{\lambda/\mu}) = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}, \quad P_{n,k} \overset{RSK}{=} \sum_{\lambda\vdash n, \lambda_1=k} \left(f^{\lambda}\right)^2.$$

Conjecture

Let

$$f_{n,k} = \sum_{\lambda\vdash n, \lambda_1=k} s_\lambda^2.$$

Then $f_{n,k}^2 - f_{n,k+1}f_{n,k-1}$ *is s-positive for* $1 \le k \le n$.

Remark. This conjecture implies the log-concavity of $P_{n,k}$.

Matchings with Given Crossing Number

Let

$$M_{2n}(q) = \sum_k M_{2n,k} q^k,$$

where $M_{2n,k}$ is the number of matchings on $[2n]$ with crossing number k.
Let

$$V_k(q) = \sum_{n \geq 0} v_k(n) \frac{q^n}{n!},$$

where $v_k(n)$ denotes the number of matchings on $[2n]$ whose crossing number is less than or equal to k.

Theorem (Grabiner-Magyar, J. Algebraic Combin. (1993); Goulden, Discrete Math. (1992))

$$V_k(q) = \det(I_{i-j}(2q) - I_{i+j}(2q))_{i,j=1}^{k}.$$

Matchings with Given Crossing Number

Note that $M_{2n,k} = v_k(n) - v_{k-1}(n)$.

$$M_2(q) = q$$

$$M_4(q) = 2q + q^2$$

$$M_6(q) = 5q + 9q^2 + q^3$$

$$M_8(q) = 14q + 70q^2 + 20q^3 + q^4$$

$$M_{10}(q) = 42q + 552q^2 + 315q^3 + 35q^4 + q^5$$

$$M_{12}(q) = 132q + 4587q^2 + 4730q^3 + 891q^4 + 54q^5 + q^6$$

$$M_{14}(q) = 429q + 40469q^2 + 71500q^3 + 20657q^4 + 2002q^5 + 77q^6 + q^7$$

Matchings with Given Crossing Number

Chen (unpublished) also made the following conjectures.

Conjecture

$M_{2n}(q)$ is log-concave for $n \geq 1$.

Conjecture

$M_{2n}(q)$ is ∞-log-concave for $n \geq 1$.

Conjecture

The polynomial sequence $\{M_{2n}(q)\}$ is strongly q-log-convex.

Conjecture

The polynomial sequence $\{M_{2n}(q)\}$ is infinitely q-log-concavity.

Schur Positivity

It is easy to see that

$$M_{2n,k} \overset{RSK}{=} \sum_{\lambda \vdash n, \lambda_1 = k} (f^\lambda).$$

Conjecture

Let

$$g_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda.$$

Then $g_{n,k}^2 - g_{n,k+1} g_{n,k-1}$ *is s-positive for* $1 \leq k \leq n$*.*

Remark. This conjecture implies the log-concavity of $M_{2n,k}$.

More Conjectures

It is well known that the polynomial $s_\lambda(1, q, q^2, \ldots, q^m)$ is unimodal for any m as a polynomial of q.
Using the theory of symmetric functions, it is easy to derive the following result

Theorem

The polynomial $h_m(\{1, q\}^n)$ is log-concave as a polynomial of q. Hence $h_\lambda(\{1, q\}^n)$ is log-concave. Similarly, the result holds for elementary symmetric functions.

Conjecture

The polynomial $s_\lambda(\{1, q\}^n)$ is log-concave as a polynomial of q.

More Conjectures

Fixing a partition λ, let

$$a_k = \sum_{|\mu|=k} s_\mu s_{\lambda/\mu}.$$

The above conjecture can be proved using the following conjecture.

Conjecture

For any $1 \leq k \leq |\lambda|$, we have $a_k^2 - a_{k+1}a_{k-1}$ is s-positive.

In particular, for $\lambda = 2^n$, we conjectured the above result holds. That is, if

$$f_k = \sum_{a=0}^{\lfloor k/2 \rfloor} s_{[2^a,1^{k-2a}]} s_{[2^{m+a-k},1^{k-2a}]},$$

then the difference

$$f_k^2 - f_{k+1}f_{k-1}$$

is s-positive.

A Related Formula

Professor R.C. King observed that a_k has an alternative expression.

Lemma (Littlewood)

Let λ, σ and τ be partitions such that $|\lambda| = |\sigma| + |\tau|$. Then

$$s_\lambda * (s_\sigma \; s_\tau) = \sum_{\mu \vdash |\sigma|} (s_\mu * s_\sigma)(s_{\lambda/\mu} * s_\tau).$$

Corollary

Let λ be a partition of weight $m = |\lambda|$. Then

$$s_\lambda * (s_{(k)} s_{(m-k)}) = \sum_{\mu \vdash k} s_\mu \; s_{\lambda/\mu} \, .$$

where (k) and $(m-k)$ are one part partitions.

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