Schur positivity arising from log-concavity problems

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Outline

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2 q-Narayana Numbers

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## Unimodality

Let \( \{a_i\}_{0 \leq i \leq m} \) be a positive sequence of real numbers.

### Definition

\( \{a_i\}_{0 \leq i \leq m} \) is **unimodal** if there exists \( k \) such that

\[
a_0 \leq \cdots \leq a_k \geq \cdots \geq a_m,
\]

and is **strictly unimodal** if

\[
a_0 < \cdots < a_k > \cdots > a_m.
\]

### Example

For fixed \( m \), \( \{\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\} \) is symmetric and unimodal. Furthermore, it is strictly unimodal if \( m \) is even.
Log-concavity

**Definition**

\{a_i\}_{0 \leq i \leq m} \text{ is log-concave if} \quad a_i^2 \geq a_{i+1}a_{i-1}

for all $1 \leq i \leq m - 1$, and is strictly log-concave if

\[ a_i^2 > a_{i+1}a_{i-1}. \]

**Remark:** A log-concave sequence is unimodal.

**Example**

For fixed $m$, \{\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}\} is strictly log-concave. While \{1, 3, 5, 9, 5, 3, 1\} is unimodal, but not log-concave.
Let $f(q) = a_0 + a_1 q + \cdots + a_m q^m$ be a polynomial with real coefficients.

**Definition**

$f(q)$ is unimodal (or strictly unimodal) if \{a_i\}_{0 \leq i \leq m} is unimodal (resp. strictly unimodal).

**Definition**

$f(q)$ is log-concave (or strictly log-concave) if \{a_i\}_{0 \leq i \leq m} is log-concave (resp. strictly log-concave).

**Example**

Let $\text{des}(\pi)$ denote the number of descents of $\pi$. The Eulerian polynomial $A_m(q) = \sum_{\pi \in \mathfrak{S}_m} q^{1+\text{des}(\pi)}$ is strictly log-concave.
Let \( \{f_i(q)\}_{0 \leq i \leq m} \) be a sequence of polynomials with real coefficients.

**Definition**

For any two polynomials \( f(q) \) and \( g(q) \) with real coefficients, define
\[ f(q) \geq_q g(q) \text{ if and only if } f(q) - g(q), \text{ as a polynomial in } q, \text{ has all nonnegative coefficients.} \]

**Definition**

\( \{f_i(q)\}_{0 \leq i \leq m} \) is **q-log-concave** if
\[ f_i(q)^2 \geq_q f_{i+1}(q)f_{i-1}(q), \quad 1 \leq i \leq m - 1, \]

and is **strongly q-log-concave** if
\[ f_i(q)f_j(q) \geq_q f_{i+1}(q)f_{j-1}(q), \quad i \geq j \geq 1. \]
q-Log-concavity

Example

The Gaussian binomial coefficients \( \{ \binom{m}{k}_q \}_{0 \leq k \leq m} \) are strongly q-log-concave.

- The q-log-concavity was conjectured by Butler (1987).
- The first proof was given by Butler (1990).
- Krattenthaler (1989) found an alternative combinatorial proof.

Remark: Usually, a q-log-concave sequence is not strongly q-log-concave.

Example

The sequence \( \{ q^2, q + q^2, 1 + 2q + q^2, 4 + q + q^2 \} \) is q-log concave but not strongly q-log concave.
**q-Log-convexity**

Based on the $q$-log-concavity, it is natural to define the $q$-log-convexity.

**Definition**

\[ \{f_i(q)\}_{0 \leq i \leq m} \text{ is } q\text{-log-convex if} \]

\[ f_i(q)^2 \leq_q f_{i+1}(q)f_{i-1}(q), \quad 1 \leq i \leq m - 1, \]

and is **strongly $q$-log-convex if**

\[ f_i(q)f_j(q) \leq_q f_{i+1}(q)f_{j-1}(q), \quad i \geq j \geq 1. \]

**Example**

The sequence

\[ \{2q + q^2 + 3q^3, q + 2q^2 + 2q^3, q + 2q^2 + 2q^3, 2q + q^2 + 3q^3\} \]

is $q$-log-convex, but not strongly $q$-log-convex.
Let $\lambda$ be a partition of $n$.

**Definition**

*The Young diagram of $\lambda$ is an array of squares in the plane justified from the top and left corner with $\ell(\lambda)$ rows and $\lambda_i$ squares in row $i$.***

Fig 1: The diagram of $(4, 3, 1)$
Semistandard Young Tableau

Definition

A semistandard Young tableau (SSYT) of shape $\lambda/\mu$ is an array $T = (T_{ij})$ of positive integers of shape $\lambda/\mu$ that is weakly increasing in every row and strictly increasing in every column.

The type of $T$ is defined as the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where $\alpha_i$ is the number of $i$’s in $T$.

```
      1  1  2  3
      2  2  4
      3
```

Fig 2: SSYT of shape $(4, 3, 1)$
If $T$ has type $\text{type}(T) = \alpha$, then we write

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots.$$ 

**Definition**

The Schur function $s_\lambda(x)$ of shape $\lambda$ is defined as the generating function

$$s_\lambda(x) = \sum_T x^T,$$

summed over all semistandard Young tableaux $T$ of shape $\lambda$. We set $s_\emptyset(x) = 1$. 

**Schur Positivity**

**Theorem**

The Schur functions $s_{\lambda}(x)$ are symmetric functions, and $\{s_{\lambda}(x) \mid \lambda \vdash n\}$ form a basis of symmetric functions of degree $n$.

**Definition**

Given a symmetric function $f(x)$, we say that it is Schur positive if all the coefficients are positive when expanding $f(x)$ in terms of Schur functions.

For a symmetric function $f(x)$, define

$$
ps_n(f) = f(1, q, \ldots, q^{n-1}),
$$

$$
ps_n^1(f) = ps_n(f)|_{q=1} = f(1^n).
$$
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The $q$-Narayana numbers, as a natural $q$-analogue of the Narayana numbers $N(n, k)$, arise in the study of $q$-Catalan numbers. The $q$-Narayana number $N_q(n, k)$ is given by

$$
N_q(n, k) = \frac{1}{[n] [k] [k - 1]} q^{k^2 - k},
$$

where we have adopted the common notation

$$
[k] := (1 - q^k)/(1 - q), \quad [k]! = [1][2] \cdots [k], \quad \begin{bmatrix} n \\ j \end{bmatrix} := \frac{[n]!}{[j]![n-j]!}
$$

for the $q$-analogues of the integer $k$, the $q$-factorial, and the $q$-binomial coefficient, respectively.
**Hook-content Formula**

A square \((i, j)\) in \(\lambda\) is the square in row \(i\) from the top and column \(j\) from the left. The hook length \(h(i, j)\), is given by \(\lambda_i + \lambda'_j - i - j + 1\). The content \(c(i, j)\) is given by \(j - i\).

**Theorem (Stanley, Studies in Applied Math. (1971))**

For any partition \(\lambda\) and \(n \geq 1\), we have

\[
\begin{align*}
\text{ps}_n(s_\lambda) &= q \sum_{k \geq 1} (k-1) \lambda_k \prod_{(i, j) \in \lambda} \frac{[n + c(i, j)]}{[h(i, j)]} \\
\text{ps}^1_n(s_\lambda) &= \prod_{(i, j) \in \lambda} \frac{n + c(i, j)}{h(i, j)}.
\end{align*}
\]
Brändén noticed that the $q$-Narayana number $N_q(n, k)$ has a Schur function expression by a specialization of the variables.

**Theorem (Brändén, Discrete Math. (2004))**

For all $n, k \in \mathbb{N}$, we have

$$N_q(n, k) = s_{(2^{k-1})}(q, q^2, \ldots, q^{n-1}).$$

Thus

$$N(n, k) = N_q(n, k)|_{q=1} = s_{(2^k)}(1^{n-1}) = ps_{n-1}1s_{(2^k)}.$$
**q-Log-concavity of $N_q(n, k)$ for Fixed $n$**

**Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))**

Given an integer $n$, the sequence $(N_q(n, k))_{k \geq 1}$ of polynomials in $q$ is strongly $q$-log-concave.

For any $k \geq l \geq 2$,

$$N_q(n, k)N_q(n, l) - N_q(n, k + 1)N_q(n, l - 1) = s(2^{k-1})s(2^{l-1}) - s(2^k)s(2^{l-2}),$$

where the Schur functions are evaluated at the variable set

$\{q, q^2, \ldots, q^{n-1}\}$.

**Theorem (Bergeron-McNamara, 2004, arXiv)**

For $k \geq 1$ and $a \geq b$, the symmetric function $s(k^a)s(k^b) - s(k^{a+1})s(k^{b-1})$ is Schur positive.

The case of $a = b$ is due to Kirillov (1984), and a different proof was given by Kleber (2001).
**q-Log-concavity of $N_q(n, k)$ for Fixed $k$**

**Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))**

Given an integer $k$, the sequence $(N_q(n, k))_{n \geq k}$ is strongly $q$-log-concave.

Proof. For any $m \geq n \geq k$, the difference

$$N_q(m, k)N_q(n, k) - N_q(m+1, k)N_q(n-1, k)$$

equals

$$q^{k-2}s_{(2k-2,1)}(X_{n-1})s_{(2k-1)}(Z) + q^{2(k-1)(m+n-1)}s_{(2k-2)}(X_{n-1})s_{(2k-1)}(Z^{-1})$$

$$+ q^{k-2} \sum_{J \subseteq (2k-2,1)} s_J(Z) \left( s_{(2k-2,1)}s_{(2k-1)} / J - s_{(2k-2,1)} / J s_{(2k-1)} \right) (X_{n-1})$$

$$+ q^{2(k-1)(m+n-1)}s_{(2k-2)}(X_{n-1})s_{(2k-2,1)}(Z^{-1})s_{(1)}(X_{n-1})$$

$$+ q^{2(k-1)(m+n-1)} \sum_{I \subseteq (2k-2)} s_I(Z^{-1}) \left( s_{(2k-2)}s_{(2k-1)} / I - s_{(2k-2)} / I s_{(2k-1)} \right) (X_{n-1})$$

where $X_r = \{q, q^2, \ldots, q^{r-1}\}$, $X_r^{-1} = \{q^{-1}, q^{-2}, \ldots, q^{-(r-1)}\}$, $Z = \{q^{n-1}, \ldots, q^{m-1}\}$ and $Z^{-1} = \{q^{1-n}, \ldots, q^{1-m}\}$.
**q-Log-concavity of** $N_q(n, k)$ **for Fixed** $k$

Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, let

\[
\begin{align*}
\lambda \vee \mu &= (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \ldots), \\
\lambda \wedge \mu &= (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots).
\end{align*}
\]

For two skew partitions $\lambda/\mu$ and $\nu/\rho$, we define

\[
\begin{align*}
(\lambda/\mu) \vee (\nu/\rho) &= (\lambda \vee \nu)/(\mu \vee \rho), \\
(\lambda/\mu) \wedge (\nu/\rho) &= (\lambda \wedge \nu)/(\mu \wedge \rho).
\end{align*}
\]

**Theorem (Lam-Postnikov-Pylyavaskyy, Amer. J. Math. (2007))**

For any two skew partitions $\lambda/\mu$ and $\nu/\rho$, the difference

\[
S(\lambda/\mu) \vee (\nu/\rho) - S(\lambda/\mu) \wedge (\nu/\rho)
\]

is Schur positive.
q-Log-concavity of $N_q(n, k)$ for Fixed $k$

**Corollary**

Let $k$ be a positive integer. If $I, J$ are partitions with $I \subseteq (2^{k-1})$ and $J \subseteq (2^{k-1}, 1)$, then both

\[ S(2^{k-1})S(2^k)/I - S(2^{k-1})/IS(2^k) \]  \hspace{1cm} (1)

and

\[ S(2^{k-1}, 1)S(2^k)/J - S(2^{k-1}, 1)/JS(2^k) \]  \hspace{1cm} (2)

are Schur positive.

Proof. For (1), take $\lambda = (2^{k-1}), \mu = I, \nu = (2^k)$ and $\rho = \emptyset$. For (2), take $\lambda = (2^{k-1}, 1), \mu = J, \nu = (2^k)$ and $\rho = \emptyset$.

Remark. The $q$-Log-Concavity of $N_q(n, k)$ for fixed $k$ follows from the above corollary.
Define the operator $\mathcal{L}$ which maps a polynomial sequence $\{f_i(q)\}_{i \geq 0}$ to a polynomial sequence given by

$$\mathcal{L}(f_i(q)) := f_i(q)^2 - f_{i-1}(q)f_{i+1}(q).$$

A sequence $\{f_i(q)\}$ is $k$-fold $q$-log-concave if $\mathcal{L}^j(f_i)$ is $q$-log-concave for $1 \leq j \leq k - 1$.

If $\{f_i(q)\}$ is $k$-fold log-concave for any $k$, then it is said to be infinitely $q$-log-concave.

**Conjecture (McNamara and Sagan, Adv. in Appl. Math. (2010))**

For fixed $k$, the Gaussian polynomials $\binom{n}{k}_{n \geq k}$ is infinitely $q$-log-concave.

Remark. For fixed $n$, they have shown that $\binom{n}{k}_k$ is not 2-fold $q$-log-concave.
Connection with a Conjecture of McNamara and Sagan

For fixed $k$, subscript the $\mathcal{L}$-operator by $n$.

$$\mathcal{L}_n\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{n-k}}{[n]} \begin{bmatrix} n \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix},$$

which are, up to a power of $q$, the $q$-Narayana numbers.

$$\mathcal{L}_n^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{3n-3k}[2]}{[n][n-1]} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k-2 \end{bmatrix}.$$

McNamara and Sagan (2010) conjectured that these polynomials are $q$-nonnegative.
Connection with a Conjecture of McNamara and Sagan

McNamara and Sagan (2010):
“It is not clear that these polynomials are $q$-nonnegative, although they must be if Conjecture 5.3 is true. Furthermore, when $q = 1$, the triangle made as $n$ and $k$ vary is not in Sloane’s Encyclopedia [24] (although it has now been submitted). We expect that these integers and polynomials have interesting, yet to be discovered, properties.”

Corollary (Chen-Wang-Yang, J. Algebraic Combin. (2010))
For fixed $k$, the Gaussian polynomials $\binom{n}{k}_{n \geq k}$ is 2-fold $q$-log-concave.
Further Result

Let \( X_n = \{q, q^2, \ldots, q^{n-1}\} \).

**Theorem (King-Yang, preprint)**

*For any partition \( \lambda \), the polynomial sequence \( \{s_\lambda(X_n)\}_{n \geq 1} \) is strongly \( q \)-log-convex. Namely, for any \( n \geq m \geq 1 \), we have*

\[
s_\lambda(X_m) s_\lambda(X_n) - s_\lambda(X_{m-1}) s_\lambda(X_{n+1}) \geq_q 0.
\]

**Proof.**

\[
s_\lambda(X_m) s_\lambda(X_n) - s_\lambda(X_{m-1}) s_\lambda(X_{n+1}) = \sum_{\rho, \mu: \lambda/\mu = h.s} q^A s_\rho(X_n/X_{m-1}) \left( s_\mu(X_{m-1}^{-1}) s_{\lambda/\rho}(X_{m-1}^{-1}) - s_\lambda(X_{m-1}^{-1}) s_{\mu/\rho}(X_{m-1}^{-1}) \right)
\]

where

\[
A = (n + m - 1)|\lambda| + f(\mu) + f(\lambda).
\]
Schur Positivity

Recall that

$$s_{(\lambda \land \nu)/(\mu \land \rho)} s_{(\lambda \lor \nu)/(\mu \lor \rho)} - s_{\lambda/\mu} s_{\nu/\rho} \geq s_0.$$  

If $\nu \subseteq \lambda$ and $\mu \subseteq \rho$ then

$$s_{\lambda/\rho} s_{\nu/\mu} - s_{\lambda/\mu} s_{\nu/\rho} \geq s_0.$$  

Setting $\mu = 0$ and then $\nu = \mu$, we find

$$s_{\lambda/\rho} s_{\mu} - s_{\lambda} s_{\mu/\rho} \geq s_0$$

for any $\mu \subseteq \lambda$. 
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**q-Log-convexity of Narayana Polynomials**

Narayana polynomial of type $A$ and $B$ are defined respectively as follows:

$$NA_n(q) = \sum_{k=0}^{n} N(n, k)q^k,$$

and

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$

**Conjecture (Liu-Wang, Adv. in Appl. Math. (2007))**

The polynomials $NA_n(q)$ form a $q$-log-convex sequence, so do $NB_n(q)$. 
q-Log-convexity of Narayana Polynomials

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

The Narayana polynomials $N_{A_n}(q)$ of type A are strongly $q$-log-convex.


The Narayana polynomials $N_{B_n}(q)$ of type B are $q$-log-convex.

Idea: $q$-log-convexity $\Rightarrow$ Schur positivity
Method: regard coefficients as specialization of symmetric functions.
Remark: Zhu (Adv. in Appl. Math., 2013) gave a simple proof of the
$q$-log-convexity of Narayana polynomials by using the recurrence
relations.
Narayana Polynomials of Type A

\[ N(n, k) = N_q(n, k)|_{q=1} = s_{(2^{k-1})}(1^{n-1}) = ps_{n-1}^1 \left( s_{(2^{k-1})} \right). \]

\[ [q^r]N_{A_{m+1}}(q)N_{A_{n-1}}(q) = \sum_{k=0}^{r-2} ps_m^1 \left( s_{(2^k)} \right) ps_{n-2}^1 \left( s_{(2^{r-2-k})} \right). \]

\[ [q^r]N_{A_{m}}(q)N_{A_{n}}(q) = \sum_{k=0}^{r-2} ps_{m-1}^1 \left( s_{(2^k)} \right) ps_{n-1}^1 \left( s_{(2^{r-2-k})} \right) \]
Narayana Polynomials of Type A

Given $a, b, m \in \mathbb{N}$ and $0 \leq i \leq m$, let

\begin{align*}
D_1(m, i, a, b) &= s_{(2i-b, 1b-a)}s_{(2m-i-1)}, \\
D_2(m, i, a, b) &= s_{(2i-b-1, 1b+2-a)}s_{(2m-i-1)}, \\
D_3(m, i, a, b) &= s_{(2i-b-1, 1b+1-a)}s_{(2m-i-1, 1)},
\end{align*}

$$D(m, i, a, b) = D_1(m, i, a, b) + D_2(m, i, a, b) - D_3(m, i, a, b).$$

The coefficient $[q^r](NA_{m+1}(q)NA_{n-1}(q) - NA_m(q)NA_n(q))$ is equal to

$$p_{n-2}^1 \left( \sum_{0 \leq a \leq b \leq d-1} p_d^1(s_{2a, 1b+1-a}) \sum_{k=0}^{r-2} D(r-2, k, a, b) \right).$$
Schur Positivity

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For any \( b \geq a \geq 0 \) and \( m \geq 0 \), the symmetric function \( \sum_{i=0}^{m} D(m, i, a, b) \) is Schur positive.

Proof is based on the case of \( a = b = 0 \).
Given a set \( S \) of positive integers, let \( \text{Par}_S(n) \) denote the set of partitions of \( n \) whose parts belong to \( S \).

Theorem (Chen-Wang-Yang, J. Algebraic Combin. (2010))

For any \( m \geq 0 \), we have

\[
\sum_{i=0}^{m} D(m, i, 0, 0) = \sum_{\lambda \in \text{Par}_{\{2,4\}}(2m-2)} s_{\lambda}.
\]  

(3)
Schur Positivity

Taking \( m = 3, 4, 5 \) and using the Maple package, we observe that

\[
\begin{align*}
\sum_{k=0}^{3} & \left( s_{2k-1} s_{2^3-k} + s_{2k-2,1} s_{2^3-k} - s_{2k-1,1} s_{2^3-k-1,1} \right) \\
& = s_{4} + s_{2,2}.
\end{align*}
\]

\[
\begin{align*}
\sum_{k=0}^{4} & \left( s_{2k-1} s_{2^4-k} + s_{2k-2,1} s_{2^4-k} - s_{2k-1,1} s_{2^4-k-1,1} \right) \\
& = s_{4,2} + s_{2,2,2}.
\end{align*}
\]

\[
\begin{align*}
\sum_{k=0}^{5} & \left( s_{2k-1} s_{2^5-k} + s_{2k-2,1} s_{2^5-k} - s_{2k-1,1} s_{2^5-k-1,1} \right) \\
& = s_{4,4} + s_{4,2,2} + s_{2,2,2,2}.
\end{align*}
\]

The proof of the above theorem mainly relies on the recurrence relations of summands \( D(m, i, 0, 0) \).

**Experiment ⇒ Observation ⇒ Proof**
Narayana Polynomials of Type $B$

When $\lambda = (1^k)$ for $k \geq 1$, the Schur function $s_{\lambda}(x)$ becomes the $k$-th elementary symmetric function $e_k(x)$, i.e.,

$$s_{(1^k)}(x) = e_k(x) = \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \quad (4)$$

$$NB_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k.$$  

$$[q^k](NB_n(q)) = \text{ps}_n^1(e_k^2).$$

$$\text{ps}_n^1(e_k) = \text{ps}_{n-1}^1(e_k + e_{k-1}).$$
Narayana Polynomials of Type B

The coefficient of $q^r$ in $NB_{n-1}(q)NB_{n+1}(q) - (NB_n(q))^2$ is given by

$$\sum_{k=0}^{r} ps_{n-1}^1(e_k)^2 ps_{n+1}^1(e_{r-k})^2 - ps_{n}^1(e_k)^2 ps_{n}^1(e_{r-k})^2.$$

⇓ apply $ps_{n}^1(e_k) = ps_{n-1}^1(e_k + e_{k-1})$ twice.

$$ps_{n-1}^1 \left( \sum_{k=0}^{r} e_k^2 (e_{r-k} + 2e_{r-k-1} + e_{r-k-2})^2 - (e_k + e_{k-1})^2 (e_{r-k} + e_{r-k-1})^2 \right).$$

⇓

$$2 ps_{n-1}^1 \left( \sum_{k=0}^{r} e_{k-1}^2 e_{r-k} + e_{k-2} e_k e_{r-k} - 2e_{k-1} e_k e_{r-k-1} e_{r-k} \right).$$
Narayana Polynomials of Type $B$


For any $r \geq 1$, we have

$$\sum_{k=0}^{r} \left( e_{k-1}e_{k-1}e_{r-k}e_{r-k} + e_{k-2}e_{k}e_{r-k}e_{r-k} - 2e_{k-1}e_{k}e_{r-k-1}e_{r-k} \right) = \sum_{\lambda} s_{\lambda},$$

where $\lambda$ sums over all partitions of $2r - 2$ of the form $(4i_4, 3^2i_3, 2^2i_2, 1^2i_1)$ with $i_1, i_2, i_3, i_4$ being nonnegative integers.

Remark. Proof relies on the Jacobi-Trudi identity.

**Theorem (The Jacobi-Trudi identity)**

Let $\lambda$ be a partition with the largest part $\leq n$ and $\lambda'$ its conjugate. Then

$$s_{\lambda}(x) = \det(e_{\lambda'_i - i + j}(x))_{i,j=1}^{n},$$

where $e_0 = 1$ and $e_k = 0$ for $k < 0$. 
Dear Boliya,

I have just seen your paper, with Chen and Wang, about Schur positivity. **Good to see something from you, I did not receive news since long.** I see that you still use ACE, but there is still the problem of recompiling it, so that in particular I can include many more libraries. ......
Dear Boliya,

I have just seen your paper, with Chen and Wang, about Schur positivity. **Good to see something from you, I did not receive news since long.** I see that you still use ACE, but there is still the problem of recompiling it, so that in particular I can include many more libraries. ......

A quick look at your article reminds me that there are many things that I did not finish, in particular in my course, the use of symmetrizing operators in symmetric function theory. As an example, I shall take your functions $D(m, r)$ p.9. It is more convenient to transpose partitions. ......
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Longest Increasing Subsequences

Let

\[ P_n(q) = \sum_k P_{n,k} q^k, \]

where \( P_{n,k} \) is the number of permutations \( \pi \) on \([n] = \{1, 2, \ldots, n\}\) such that the length of the longest increasing subsequences of \( \pi \) equals \( k \).

**Theorem (Baik-Deift-Johansson, J. Amer. Math. Soc. (1999))**

The limiting distribution of the coefficients of \( P_n(q) \) is the Tracy-Widom distribution.

The numbers \( P_{n,k} \) can be computed by Gessel’s theorem. Let \( \mathfrak{S}_n \) be the symmetric group on \([n]\), and let \( \text{is}(\pi) \) be the length of the longest increasing subsequences of \( \pi \).
Longest Increasing Subsequences

Define

\[ u_k(n) = \#\{w \in \mathfrak{S}_n : \text{is}(w) \leq k\}, \quad (5) \]

\[ U_k(q) = \sum_{n \geq 0} u_k(n) \frac{q^{2n}}{n!^2}, \quad k \geq 1, \quad (6) \]

\[ l_i(2q) = \sum_{n \geq 0} \frac{q^{2n+i}}{n!(n+i)!}, \quad i \in \mathbb{Z}. \quad (7) \]

**Theorem (Gessel, J. Combin. Theory, Ser. A (1990))**

\[ U_k(q) = \det(l_i-j(2q))_{i,j=1}^k. \]
Longest Increasing Subsequences

Note that $P_{n,k} = u_k(n) - u_{k-1}(n)$ for $n \geq 1$.

\[
P_1(q) = q,
\]
\[
P_2(q) = q + q^2,
\]
\[
P_3(q) = q + 4q^2 + q^3,
\]
\[
P_4(q) = q + 13q^2 + 9q^3 + q^4,
\]
\[
P_5(q) = q + 41q^2 + 61q^3 + 16q^4 + q^5,
\]
\[
P_6(q) = q + 131q^2 + 381q^3 + 181q^4 + 25q^5 + q^6,
\]
\[
P_7(q) = q + 428q^2 + 2332q^3 + 1821q^4 + 421q^5 + 36q^6 + q^7.
\]
Definitions

q-Narayana Numbers

Narayana polynomials

Some Open Problems

Longest Increasing Subsequences

Conjecture

\[ P_n(q) \text{ is log-concave for } n \geq 1. \]

Conjecture

\[ P_n(q) \text{ is } \infty\text{-log-concave for } n \geq 1. \]

Conjecture

The polynomial sequence \( \{P_n(q)\} \) is strongly \( q \)-log-convex.

Conjecture

The polynomial sequence \( \{P_n(q)\} \) is infinitely \( q \)-log-convex.

These conjectures were proposed by W.Y.C. Chen (unpublished).
**Longest Increasing Subsequences**

Let $f^{\lambda/\mu}$ denote the number of standard Young tableaux of shape $\lambda/\mu$. The exponential specialization is a homomorphism $\text{ex} : \Lambda \to \mathbb{Q}[t]$, defined by $\text{ex}(p_n) = t\delta_{1n}$, where $p_n$ is the $n$-th power sum. Let $\text{ex}_1(f) = \text{ex}(f)_{t=1}$, provided this number is defined. It is known that

$$\text{ex}_1(s_{\lambda/\mu}) = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}, \quad P_{n,k}^\text{RSK} = \sum_{\lambda \vdash n, \lambda_1 = k} (f^{\lambda})^2.$$

**Conjecture**

Let

$$f_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_{\lambda}^2.$$  

Then $f_{n,k}^2 - f_{n,k+1}f_{n,k-1}$ is $s$-positive for $1 \leq k \leq n$.

Remark. This conjecture implies the log-concavity of $P_{n,k}$. 
Matchings with Given Crossing Number

Let

\[ M_{2n}(q) = \sum_k M_{2n,k} q^k, \]

where \( M_{2n,k} \) is the number of matchings on \([2n]\) with crossing number \( k \).

Let

\[ V_k(q) = \sum_{n \geq 0} v_k(n) \frac{q^n}{n!}, \]

where \( v_k(n) \) denotes the number of matchings on \([2n]\) whose crossing number is less than or equal to \( k \).

**Theorem (Grabiner-Magyar, J. Algebraic Combin. (1993); Goulden, Discrete Math. (1992))**

\[ V_k(q) = \det(l_{i-j}(2q) - l_{i+j}(2q))_{i,j=1}^k. \]
Matchings with Given Crossing Number

Note that \( M_{2n,k} = v_k(n) - v_{k-1}(n) \).

\[
\begin{align*}
M_2(q) &= q \\
M_4(q) &= 2q + q^2 \\
M_6(q) &= 5q + 9q^2 + q^3 \\
M_8(q) &= 14q + 70q^2 + 20q^3 + q^4 \\
M_{10}(q) &= 42q + 552q^2 + 315q^3 + 35q^4 + q^5 \\
M_{12}(q) &= 132q + 4587q^2 + 4730q^3 + 891q^4 + 54q^5 + q^6 \\
M_{14}(q) &= 429q + 40469q^2 + 71500q^3 + 20657q^4 + 2002q^5 + 77q^6 + q^7
\end{align*}
\]
**Matchings with Given Crossing Number**

Chen (unpublished) also made the following conjectures.

<table>
<thead>
<tr>
<th>Conjecture</th>
<th>( M_{2n}(q) ) is log-concave for ( n \geq 1 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjecture</td>
<td>( M_{2n}(q) ) is ( \infty )-log-concave for ( n \geq 1 ).</td>
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</tr>
</tbody>
</table>
Schur Positivity

It is easy to see that

\[
M_{2n,k}^{\text{RSK}} = \sum_{\lambda \vdash n, \lambda_1 = k} (f^\lambda).
\]

**Conjecture**

Let

\[
g_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda.
\]

Then \( g_{n,k}^2 - g_{n,k+1}g_{n,k-1} \) is s-positive for \( 1 \leq k \leq n \).

Remark. This conjecture implies the log-concavity of \( M_{2n,k} \).
More Conjectures

It is well known that the polynomial $s_\lambda(1, q, q^2, \ldots, q^m)$ is unimodal for any $m$ as a polynomial of $q$.

Using the theory of symmetric functions, it is easy to derive the following result

**Theorem**

The polynomial $h_m(\{1, q\}^n)$ is log-concave as a polynomial of $q$. Hence $h_\lambda(\{1, q\}^n)$ is log-concave. Similarly, the result holds for elementary symmetric functions.

**Conjecture**

The polynomial $s_\lambda(\{1, q\}^n)$ is log-concave as a polynomial of $q$. 
More Conjectures

Fixing a partition $\lambda$, let

$$a_k = \sum_{|\mu|=k} s_\mu s_{\lambda/\mu}.$$

The above conjecture can be proved using the following conjecture.

**Conjecture**

*For any $1 \leq k \leq |\lambda|$, we have $a_k^2 - a_{k+1}a_{k-1}$ is s-positive.*

In particular, for $\lambda = 2^n$, we conjectured the above result holds. That is, if

$$f_k = \sum_{a=0}^{\lfloor k/2 \rfloor} S[2^a, 1^{k-2a}] S[2^m + a - k, 1^{k-2a}],$$

then the difference

$$f_k^2 - f_{k+1}f_{k-1}$$

is s-positive.
Professor R.C. King observed that $a_k$ has an alternative expression.

**Lemma (Littlewood)**

Let $\lambda$, $\sigma$ and $\tau$ be partitions such that $|\lambda| = |\sigma| + |\tau|$. Then

$$s_\lambda * (s_\sigma s_\tau) = \sum_{\mu \vdash |\sigma|} (s_\mu * s_\sigma)(s_{\lambda/\mu} * s_\tau).$$

**Corollary**

Let $\lambda$ be a partition of weight $m = |\lambda|$. Then

$$s_\lambda * (s(k)s(m-k)) = \sum_{\mu \vdash k} s_\mu s_{\lambda/\mu}.$$

where $(k)$ and $(m - k)$ are one part partitions.
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