Rigidity Results for Elliptic PDEs

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Workshop on New Trends in Nonlinear Elliptic Equations
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Outline

1. Allen-Cahn equation
2. Lane-Emden equation

Ennio De Giorgi

Research Area:

- Minimal Surfaces and Bernstein's problem
- Geometric Measure Theory
- Regularity of Solutions of Elliptic Equations (Hilbert's 19th problem with John Nash 56-57)
- Γ-Convergence Theory

Awards:

- Caccioppoli Prize (1960)
- Wolf Prize (1990)

Figure: 1928-1996

Quote:
If you can't prove your theorem, keep shifting parts of the conclusion to the assumptions, until you can.

Allen-Cahn equation
Lane-Emden equation

Allen-Cahn Equation:

$$-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n.$$

Euler-Lagrange equation for the energy functional:

$$E(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1-u^2)^2$$

$u = 1$ and $u = -1$ are global minimizers of the energy and representing, in the gradient theory of phase transitions, two distinct phases of a material.

$$F(u) = -\frac{1}{4}(1-u^2)^2$$

is called "double-well potential":

$$F(+1) = F(-1) = 0 \text{ and } F(u) \neq 0 \text{ if } u \neq \pm 1$$

Example: In dimension one $w(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ solves the equation and $w' > 0$ and w connects -1 to 1 that is $w(\pm\infty) = \pm 1$.

Ennio De Giorgi (1978) expected that the interface between the phases $u = 1$ and $u = -1$ has to approach a minimal surface.
Bernstein's Conjecture: Any minimal surface in $\mathbb{R}^n$ must be a hyperplane.
Equivalent to any entire solution of the form $x_n = F(x_1, \cdots, x_{n-1})$ of

$$\mathrm{div}\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = 0 \text{ in } \mathbb{R}^{n-1}$$

must be a linear function that is $F(x_1, \cdots, x_{n-1}) = a \cdot (x_1, \cdots, x_{n-1}) + b$ for some $a \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$.
True for $n \leq 8$: Bernstein (1910 Math Z), Fleming (1962 Math Palermo), De Giorgi (1965 Annali Pisa), Almgren (1966 Annals Math), Simons (1968 Annals Math).
False for $n \geq 9$: counterexample by Bombieri-De Giorgi-Giusti (1969 Invent Math).
This led him to state his conjecture.

Allen-Cahn equation
Lane-Emden equation

De Giorgi's Conjecture (1978): Suppose that u is bounded and monotone (in one direction) solution of the Allen-Cahn equation

$$-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n$$

Then, at least for $n \leq 8$, solutions are one-dimensional, i.e. $u(x) = u^*((x - \nu) \cdot p)$ for some ν, p.
$\Longrightarrow u(x) = \tanh\left(\frac{x.a-b}{\sqrt{2}}\right)$ where $b \in \mathbb{R}$, $|a| = 1$ and $a_n > 0$.

- For $n = 2$ by Ghoussoub-Gui (1997 Math Ann)
- For $n = 3$ by Ambrosio-Cabré (2000 J. AMS)
- For $n = 4, 5$, if u is anti-symmetric, by Ghoussoub-Gui (2003 Annals Math)
- For $4 \leq n \leq 8$, if u satisfies the additional (natural) assumption

$$\lim_{x_n \to \pm\infty} u(\mathbf{x}', x_n) \to \pm 1.$$ Savin (2003 Annals Math)

  2nd Proof: Wang (2014 Arxiv)
- Counterexample for $n \geq 9$, by del Pino-Kowalczyk-Wei (2008 Annals Math)

Note: In lower dimensions, it is proved for any nonlinearity $-\Delta u = f(u)$. For $n = 2$ the same paper and for $n = 3$ by Alberti-Ambrosio-Cabré (2001 Acta Appl. Math.)

Observations to prove the De Giorgi's conjecture

Consider PDE:

$$-\Delta u = f(u) \qquad x \in \mathbb{R}^n.$$

1. Monotonicity $\Longrightarrow$ Pointwise Stability $\Longleftrightarrow$ Stability.
   - Pointwise Stability: $\exists \phi > 0$ that
   $$-\Delta \phi = f'(u)\phi \quad \text{in} \quad \mathbb{R}^n.$$
   - Stability (or Stability Inequality): if the second variation of the energy is non-negative:
   $$\int f'(u)\zeta^2 \leq \int |\nabla \zeta|^2 \qquad \forall \zeta \in C_c^2(\mathbb{R}^n)$$
2. Set $\phi := \partial_{x_n} u$ and $\psi := \partial_{x_i} u$, then the quotient $\sigma = \frac{\psi}{\phi}$ satisfies a linear equation $\mathrm{div}(\phi^2 \nabla \sigma) = 0$.

   It is shown by Berestycki-Caffarelli-Nirenberg, Ambrosio-Cabre and Ghoussoub-Gui in 97-98 that if $\phi > 0$ and
   $$\int_{B_R} \phi^2 \sigma^2 < R^2, \qquad \forall R > 1$$
   then $\sigma = 0$.

Note: Is this optimal? Consider R^{a_n} then

- $a_n < n$ where $n \geq 3$. Barlow (1998 Can J Math)
- $a_n < 2 + 2\sqrt{n-1}$ when $n \geq 7$. Ghoussoub-Gui (1998 Math Ann)
- IF $a_n \geq n - 1$ then conjecture would establish in n-D.

Allen-Cahn equation
Lane-Emden equation

- If the limit

$$\lim_{x_n \to \pm\infty} u(x', x_n) \to \pm 1 \quad \text{in} \quad \mathbb{R}^{n-1}$$

is uniform $\Longrightarrow$ called Gibbon's conjecture and proved (1999) in all dimensions by Farina (Mat e Appli), Barlow-Bass-Gui (CPAM), Berestycki-Hamel-Monneau (Duke Math)

- Stability Conjecture: Let u be a bounded stable solution of Allen-Cahn equation. Then the level sets $u = \lambda$ are all hyperplanes.
    - True in $n = 2$ by Ambrosio-Cabre and Ghoussoub-Gui.
    - False in $n = 8$ by Pacard-Wei (2013 JFA)
    - Classification is open in other dimensions.
- Fractional Laplacian case: $(-\Delta)^s u = f(u)$ and $s \in (0, 1)$
    - Existence when $n = 1$ by Cabre-Sire (2009 Annales Poincare).
    - For any s when $n = 2$ by Sire-Valdinoci (2009 J FA).
    - For any $s \in [1/2, 1)$ when $n = 3$ by Cabre-Cinti (2012 DCDS).
    - Open for other cases.

The proof strongly relies on the extension function given by Caffarelli-Silvestre (2007 CPDE), i.e.

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla u_e) & = & 0 \ \text{in} \ \mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^n, y > 0\}, \\ -\lim_{y \to 0} y^{1-2s} \partial_y u_e & = & k_s f(u_e) \ \text{in} \ \partial \mathbb{R}^{n+1}_+, \end{cases}$$

- Modica's estimate:

$$|\nabla u|^2 \leq 2F(u) \text{ for bounded solutions of } \Delta u = f(u) \text{ in } \mathbb{R}^n$$

when $F' = f$ and $F \geq 0$ by Modica (1980 CPAM).
Ex.: $|\nabla u|^2 \leq \frac{1}{2}(u^2 - 1)^2$ for the Allen-Cahn equation.
Ex.: $|\nabla u|^2 \leq 2(1 - \cos u)$ for bd solutions of $\Delta u = \sin u$

- Monotonicity Formula:

$$\Gamma_R = \frac{1}{R^{n-1}} \int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u)$$

is nondecreasing in R.

Pointwise inequalities in $\mathbb{R}^n$; lower order operators

- Semilinear PDEs: $\Delta u = f(u) \xrightarrow{\text{Modica (1980 CPAM)}} |\nabla u|^2 \leq 2F(u)$ in $\mathbb{R}^n$.
  How: Define $W(x) = |\nabla u(x)|^2 - 2F(u(x))$ then

$$|\nabla u|^2 \Delta W \geq \frac{1}{2}|\nabla W|^2 + 2f(u)\nabla u \cdot \nabla W \quad \text{in} \quad \mathbb{R}^n$$

- Quasilinear PDEs: $\mathrm{div}(\Phi'(|\nabla u|^2)\nabla u) = f(u)$ in $\mathbb{R}^n$ for some ϕ
  $\xrightarrow{\text{Caffarelli-Garofalo-Segala (1994 CPAM)}} 2\Phi'(|\nabla u|^2)|\nabla u|^2 - \Phi(|\nabla u|^2) \leq 2F(u)$.
  EX.:
  - The p-Laplacian operator: $\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u)$ then

  $$|\nabla u|^p \leq \frac{p}{p-1}F(u).$$

  - The prescribed mean curvature operator: $\mathrm{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = f(u)$ then

  $$\frac{\sqrt{1+|\nabla u|^2} - 1}{\sqrt{1+|\nabla u|^2}} \leq F(u)$$

  How: Work with the difference function!
- Nonlocal PDEs: $(-\Delta)^s u = f(u) \xrightarrow{\text{Cabre-Sire (2014 Annales Poincare)}}$

$$\int_0^y t^{1-2s}[(\partial_x u_e(x,t))^2 - (\partial_y u_e(x,t))^2]dt \leq 2F(u(x,0))$$

in $\mathbb{R}^n$ known only for $n = 1$. For any $y \geq 0$ and $0 < s < 1$.

What do we know about fourth order PDEs?

- PDE: $\Delta^2 u = f(u) \to$ Not known in $\mathbb{R}^n$.
- PDE: $\Delta^2 u = u^p, p > 1 \xrightarrow{\text{Wei-Xu (1991 Math Ann)}} -\Delta u > 0$ in $\mathbb{R}^n$.
  Application: Classifications for $(-\Delta)^m u = u^p$, $1 \leq m \in \mathbb{Z}$ in $\mathbb{R}^n$.
  How: Using ODE methods i.e. the average function on the Sphere.
- PDE: $\Delta^2 u = u^p, p > 1 \xrightarrow{\text{Souplet (2009 Adv Math)}} -\Delta u \geq \sqrt{\frac{2}{p+1}} u^{\frac{p+1}{2}}$ in $\mathbb{R}^n$.
  Application: Lane-Emden conjecture in four dimensions.
  How: $W(x) = \Delta u(x) + \sqrt{\frac{2}{p+1}} u^{\frac{p+1}{2}}$ then

$$\begin{aligned}\Delta W(x) &= u^p + \sqrt{\frac{p+1}{2}} u^{\frac{p-1}{2}} \Delta u + \sqrt{\frac{p+1}{2}} \left(\frac{p-1}{2}\right) |\nabla u|^2 u^{\frac{p-3}{2}} \\ &= \sqrt{\frac{p+1}{2}} u^{\frac{p-1}{2}} W(x) + Brown\end{aligned}$$

Note: $|\nabla u|^2 u^{\frac{p-3}{2}} = u^{\frac{p-1}{2}} \frac{|\nabla u|^2}{u}$. It seems $\frac{|\nabla u|^2}{u}$ is comparable to Δu.

Moser (1961 CPAM) developed an iteration argument in the regularity theory. Define a sequence of functions $\{W_k(x)\}_{k=-1}$

$$W_k(x) := \Delta u + \alpha_k |\nabla u|^2 (u+\epsilon)^{-1} + \beta_k u^{\frac{p+1}{2}}$$

where $\alpha_{k+1} \geq \alpha_k$ and $\beta_{k+1} \geq \beta_k$ and $\alpha_{-1} = \beta_{-1} = 0$.
Iteration argument:

- $W_{-1}(x) \leq 0$.
- $W_0(x) \leq 0$ when $\alpha_0 = 0$ and $\beta_0 = \sqrt{\frac{2}{p+1}}$.
- Suppose that $W_k \leq 0$ then W_{k+1} satisfies

$$\begin{aligned} & \Delta w_{k+1} - \beta_{k+1} \frac{(p+1)}{2} u^{\frac{p-1}{2}} w_{k+1} \\ & -2\alpha_{k+1} \nabla u \cdot \nabla w_{k+1} (u+\epsilon)^{-1} + \alpha_{k+1} w_{k+1} |\nabla u|^2 (u+\epsilon)^{-2} \\ \geq \quad & I_k(x, u, W_{k+1}, \alpha_k, \beta_k) \end{aligned}$$

  What we lose in $\geq$: Going from Hessian to Laplacian, same as the Modica's proof.

Conclusion: $\Delta^2 u = u^p, p > 1$ *then* $-\Delta u \geq \sqrt{\frac{2}{p+1}} u^{\frac{p+1}{2}} + \frac{2}{n-4} \frac{|\nabla u|^2}{u}$ in $\mathbb{R}^n$.
Fazly-Wei-Xu (2015 Analysis & PDE)

Allen-Cahn equation
Lane-Emden equation

To extend the De Giorgi's conjecture to systems, what is the right system? Consider the gradient system:

$$\Delta u = \nabla H(u) \text{ in } \mathbb{R}^n,$$

where $u : \mathbb{R}^n \to \mathbb{R}^k$ is bounded and $H \in C^2(\mathbb{R}^k)$.
Euler-Lagrange equation for the energy functional:

$$E(u) = \frac{1}{2}\int \sum_{i=1}^{k} |\nabla u_i|^2 + \int (H(u) - \inf_u H(u))$$

Phase Transitions: Minimum points of H are global minimizers of the energy and representing distinct phases of k materials.
Example: For $k = 2$ and $H(u, v) = u^2 v^2$ the global minimizers are $u = 0$ and $v = 0$.
We need Monotonicity and Stability concepts for systems.

- *H*-Monotone:
  1. For every $i \in \{1, \cdots, k\}$, u_i is strictly monotone in the x_n-variable (i.e., $\partial_{x_n} u_i \neq 0$).
  2. For $i < j$, we have
  $$H_{u_i u_j} \partial_{x_n} u_i(x) \partial_{x_n} u_j(x) < 0 \text{ for all } x \in \mathbb{R}^n.$$
  This condition implies a combinatorial assumption on $H_{u_i u_j}$ and we call such a system orientable.
- Pointwise Stability: $\exists (\phi_i)_{i=1}^k$ non sign changing
$$\Delta \phi_i = \sum_j H_{u_i u_j} \phi_j$$
and $H_{u_i u_j} \phi_j \phi_i \leq 0$ for $1 \leq i < j \leq k$.
- Stability (or Stability Inequality):
$$\sum_i \int_{\mathbb{R}^n} |\nabla \zeta_i|^2 + \sum_{i,j} \int_{\mathbb{R}^n} H_{u_i u_j} \zeta_i \zeta_j \geq 0,$$
for every $\zeta_i \in C_c^1(\mathbb{R}^n), i = 1, \cdots, k$.

Orientable systems: H-Monotonicity $\Longrightarrow$ Pointwise Stability $\Longleftrightarrow$ Stability.

Theorem (Fazly-Ghoussoub, Calc PDE 2013)

Suppose $u = (u_i)_{i=1}^k$ is a bounded H-monotone solution, then for $n \leq 3$ each component u_i must be one-dimensional. Moreover, ∇u_i and ∇u_j are parallel.

Note: Gradients are parallel via geometric Poincaré inequality:

$$\sum_i \int_{\mathbb{R}^n} |\nabla u_i|^2 |\nabla \eta_i|^2 \geq \sum_i \int_{\mathbb{R}^n \cap \{|\nabla u_i| \neq 0\}} \left(|\nabla u_i|^2 \mathcal{A}_i^2 + |\nabla_T |\nabla u_i||^2 \right) \eta_i^2$$
$$+ \sum_{i \neq j} \int_{\mathbb{R}^n} \left(\nabla u_i \cdot \nabla u_j \eta_i^2 - |\nabla u_i||\nabla u_j| \eta_i \eta_j \right) H_{u_i u_j},$$

When $m = 1$, inequality by Sternberg-Zumbrun (1998 ARMA) and applied by Farina-Sciunzi-Valdinoci (2008 Ann. Pisa) and Cabre (2010 CPAM).
Remarks:

- Alama, Bronsard, Gui (1997 Calc PDE) constructed 2D solutions $u : \mathbb{R}^2 \to \mathbb{R}^2$ that are not H-monotone. H-monotonicity is a crucial assumption!.
- Brendan Pass (2011 PhD Thesis) observed a similar concept called "compatible cost" in multi-marginal optimal transport. Equivalent: Ghoussoub-Pass (2014 CPDE)

Set $\phi_i := \partial_{x_n} u_i$ and $\psi_i := \nabla u_i \cdot \eta$ for $\eta = (\eta', 0) \in \mathbb{R}^{n-1} \times \{0\}$ then $\sigma_i := \frac{\psi_i}{\phi_i}$ satisfies a linear equation

$$\mathrm{div}(\phi_i^2 \nabla \sigma_i) + \sum_{j=1}^{k} h_{i,j}(x)(\sigma_i - \sigma_j) = 0 \ \text{ in } \ \mathbb{R}^n$$

where $h_{i,j}(x) = H_{u_i u_j} \phi_i \phi_j$.

- Linear Liouville Theorem: If σ_i satisfies the above, $\phi_i > 0$, $h_{i,j} = h_{j,i} < 0$ and

$$\sum_{i=1}^{k} \int_{B_{2R} \setminus B_R} \phi_i^2 \sigma_i^2 < CR^2, \quad \forall R > 1$$

$\Longrightarrow$ then each σ_i is constant.

- Energy estimates:

$$\text{Bounded stable} \Longrightarrow \sum_{i=1}^{k} \int_{B_R} |\nabla u_i|^2 \leq E_R(u) \leq CR^{n-1}.$$

Optimality not known when $k > 1$.

- Modica's estimates does not hold in general, by Farina (2004 J FA)

$$\sum_{i=1}^{k} |\nabla u_i|^2 \leq 2H(u) \quad \textit{Nope}!$$

However a Hamiltonian identity given by Gui (2008 J FA)

$$\int_{R^{n-1}} \left[\sum_{i=1}^{k} \left(|\nabla_{x'} u_i|^2 - |\partial_{x_n} u_i|^2 \right) - 2H(u(x', x_n)) \right] dx' = C \quad \text{for} \quad x_n \in \mathbb{R}$$

- Here is a Hamiltonian identity that is a counterpart of Caffarelli-Garofalo-Segala pointwise inequality.

$$\int_{\mathbb{R}^{n-1}} \left[\sum_{i=1}^{m} \left(\Phi\left(|\nabla u_i|^2 \right) - 2\Phi'\left(|\nabla u_i|^2 \right) |\partial_{x_n} u_i|^2 \right) - 2H(u(x', x_n)) \right] dx' \equiv C$$

for $x_n \in \mathbb{R}$ by Fazly (2015 Arxiv)

- When is $\Gamma_R = \frac{E_R(u)}{R^{n-1}}$ increasing? Not known when $k > 1$. $\tilde{\Gamma}_R = \frac{E_R(u)}{R^{n-2}}$ is nondecreasing.

- Fractional system $-(-\Delta)^s u = \nabla H(u)$ when $n=2$ and $s \in (0,1)$ and $n=3$ and $1/2 \le s < 1$ by Fazly-Sire (2015 CPDE).
  - $\tilde{\Gamma}_R = \frac{E_R(u)}{R^{n-2s}}$ is increasing where
  $$E_R(u) = \frac{1}{2}\int_{B_R \cap \mathbb{R}^{n+1}_+} \sum_{i=1}^{k} y^{1-2s} |\nabla u_i|^2 d\mathbf{x}dy + \int_{B_R \cap \partial\mathbb{R}^{n+1}_+} H(u) d\mathbf{x}$$
  (Idea: Pohozaev Identity)
  - If $u = u(|\mathbf{x}|, y)$ then $I_r(u)$ is nondecreasing in r where
  $$I_r(u) = \sum_{i=1}^{k} \int_0^\infty y^{1-2s} \left[(\partial_r u_i)^2 - (\partial_y u_i)^2\right] dy + 2H(u(r,0))$$
  - Let $n=1$ and $\lim_{x\to\infty} u = \alpha$ then for $x \in \mathbb{R}$
  $$\sum_{i=1}^{k} \int_0^\infty y^{1-2s} \left[(\partial_x u_i)^2 - (\partial_y u_i)^2\right] dy + 2H(u(x,0)) = 2H(\alpha).$$
- For the case $k=2$ and $H(u,v) = \frac{1}{2}u^2v^2$ and $\Delta u = \nabla H(u)$ then
  - there exists 1-D solutions of the form $u(x-x_0) = v(x_0 - x)$. Berestycki-Lin-Wei-Zhao (2013 ARMA)
  - 1-D solution is unique. Berestycki-Terracini-Wang-Wei (2013 Adv Math)
- For the case $k=2$ and $H(u,v) = \frac{1}{2}u^2v^2$ and $-(-\Delta)^s u = \nabla H(u)$ then
  - there exists a unique 1-D solution. Wang-Wei (2015 Math Ann)

Nonnegative solutions and $p > 1$:

$$-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^n$$

Theorem (Gidas and Spruck, 1980)

Let $n \geq 3$ and p be under the Sobolev exponent, $1 < p < \frac{n+2}{n-2} =: p^*(n)$. Then $u = 0$.

Critical case $p = \frac{n+2}{n-2}$:

- Gidas-Ni-Nirenberg (1981 MAA) proved that all solutions with $u(x) = O(|x|^{2-n})$ are radially symmetric about some $x_0 \in \mathbb{R}^n$ and of the form

$$u(x) = C_n \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2}{2}}$$

  where $C_n = (n(n-2))^{\frac{2-n}{4}}$, $\lambda > 0$ and some $x_0 \in \mathbb{R}^n$.
- Caffarelli-Gidas-Spruck (1989 CPAM) removed the condition.
- Chen and Li (1991 Duke Math) via moving plane methods.

Note:

- Fourth order case: Wei-Xu (1999 Math Ann). Here $p^*(n) := \frac{n+4}{n-4}$.
- Fractional case: YanYan Li (2004 JEMS) and Chen-Li-Ou (2006 CPAM). Here $p_s^*(n) := \frac{n+2s}{n-2s}$.

Allen-Cahn equation
Lane-Emden equation

Stable solutions and $p > 1$:

$$(-\Delta)^s u = |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^n$$

there exists $p_s^{**}(n)$, called Joseph-Lundgren exponent, such that for $1 < p < p_s^{**}(n)$, $u = 0$.

- For $s = 1$ Farina (2007 J Math Pure Appl) where

$$p_1^{**}(n) = \begin{cases} \infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11, \end{cases}$$

- For $s = 2$ Davila-Dupagine-Wang-Wei (2014 Adv Math) where

$$p_2^{**}(n) = \begin{cases} \infty & \text{if } n \leq 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13, \end{cases}$$

- For $0 < s < 1$ Davila-Dupagine-Wei (2015 Trans AMS)
- For $1 < s < 2$ Fazly-Wei (2015 Amer J Math) where $p_s^{**}(n)$ can be found from

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}$$

Optimal. For $p \geq p_s^{**}(n)$ there is a stable solution that is radially symmetric w.r.t. some point.

- Case $s=1$: Evans (91 ARMA) and Pacard (93 Manuscripta Math).

$$E(x_0,r) := r^{-n+2\frac{p+1}{p-1}} \int_{B_r(x_0)} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) + \frac{r^{-1-n+2\frac{p+1}{p-1}}}{p-1} \int_{\partial B_r(x_0)} |u|^2$$

- Case $1<s<2$, $E(x_0,r)$ is the following

$$\begin{aligned}
& r^{2s\frac{p+1}{p-1}-n}\left(\int_{\mathbb{R}^{n+1}_+\cap B_r(x_0)} \frac{1}{2} y^{3-2s}|\Delta_b u_e|^2 - \frac{1}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_r(x_0)} u_e^{p+1}\right)\\
& - Cr^{-3+2s+\frac{4s}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap \partial B_r(x_0)} y^{3-2s} u_e^2\\
& - C\partial_r\left[r^{\frac{4s}{p-1}+2s-2-n}\int_{\mathbb{R}^{n+1}_+\cap \partial B_r(x_0)} y^{3-2s} u_e^2\right]\\
& + \frac{1}{2} r^3\partial_r\left[r^{\frac{4s}{p-1}+2s-3-n}\int_{\mathbb{R}^{n+1}_+\cap \partial B_r(x_0)} y^{3-2s}\left(\frac{2s}{p-1}r^{-1}u+\partial_r u_e\right)^2\right]\\
& + \frac{1}{2}\partial_r\left[r^{2s\frac{p+1}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap \partial B_r(x_0)} y^{3-2s}\left(|\nabla u_e|^2 - |\partial_r u_e|^2\right)\right]\\
& + \frac{1}{2} r^{2s\frac{p+1}{p-1}-n-1}\int_{\mathbb{R}^{n+1}_+\cap \partial B_r(x_0)} y^{3-2s}\left(|\nabla u_e|^2 - |\partial_r u_e|^2\right)
\end{aligned}$$

where $\Delta_b u_e := y^{-b}\,\mathrm{div}(y^b \nabla u_e)$. Extension function: Ray Yang (2013 Arxiv).

- Monotoniciy Formula implies $u = r^{-\frac{2s}{p-1}}\psi(\theta)$ that is called Homogenous Solution.

Goal: $\psi \equiv 0$ where $1 < p < p_s^{**}(n)$. How?

- Step 1. From PDE:

$$A_{n,s}\int_{\mathbb{S}^{n-1}}\psi^2 + \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} K_{\frac{2s}{p-1}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2 = \int_{\mathbb{S}^{n-1}}\psi^{p+1}$$

  where $A_{n,s}$ is explicitly known and $K_\alpha(<\theta,\sigma>)$ is decreasing in α for $p > p_s^*(n)$.

- Step 2. From Stability: Test on $r^{-\frac{n-2s}{2}}\psi(\theta)\eta_\epsilon(r)$ for appropriate $\eta_\epsilon(r)$ to get

$$\Lambda_{n,s}\int_{\mathbb{S}^{n-1}}\psi^2 + \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2 \geq p\int_{\mathbb{S}^{n-1}}\psi^{p+1}$$

  where $\Lambda_{n,s}$ is the Hardy constant.

- Note that $K_{\frac{n-2s}{2}} < K_{\frac{2s}{p-1}}$ for $p > p_s^*(n)$. If $\Lambda_{n,s} < pA_{n,s}$ then $\psi = 0$.

Wei's Conjecture: If $p_1^{**}(n) \leq p < p_1^{**}(n-1)$, all stable solutions are radially symmetric?

Note: For $\frac{n+1}{n-3} < p < p_1^{**}(n-1)$ there are unstable nonradial solutions.

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Thank you for your attention.