Infinite-time bubbling in the critical heat equation: the role of Green's function

Monica Musso

Universidad Católica de Chile

The critical semilinear heat equation

$$\frac{\partial u}{\partial t} = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0,\infty), \qquad u > 0 \quad \text{in } \Omega \times (0,\infty)$$

$$\text{(P)} \qquad u = 0 \quad \text{on } \partial\Omega \times (0,\infty).$$

Ω smooth, bounded domain in $\mathbb{R}^n$, $n \geq 3$.

Our purpose: to study solutions that are smooth, globally defined in time and unbounded, namely with **infinite-time blow-up.**

The equation

$$\frac{\partial u}{\partial t} = \Delta u + u^{\frac{n+2}{n-2}}$$

is a model for various geometric flows where **bubbling phenomena** appears: family of steady states with equal energy depending on a concentration parameter:

$$U_\mu(x-\xi) = \alpha_n \left(\frac{\mu}{\mu^2 + |x-\xi|^2} \right)^{\frac{n-2}{2}}$$

e.g. Harmonic map flow, Yamabe flow, 2d-Ricci flow.

We construct solutions which as $t \to \infty$ look like

$$u(x,t) \sim \sum_{j=1}^{k} U_{\mu_j(t)}(x - \xi_j(t)), \quad \mu_j(t) \to 0$$

The more general problem: Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$.

$$(\mathrm{Q})_p \qquad \frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \quad \text{in } \mathcal{D} \times (0,\infty), \quad p > 1$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \times (0,\infty).$$

• Fujita 1966.

• The role of the critical exponent $p = p_S := \frac{n+2}{n-2}$ for threshold phenomena has been widely considered after Pohozaev 1965: no nontrivial stationary states exist if $\mathcal{D}$ is star-shaped.

Let φ be a positive function, $u_\lambda(x,t)$ the solution of $(\mathrm{Q})_p$ with $u(x,0) = \lambda\varphi(x)$.

$$\lambda_* = \sup\{\lambda > 0 \ / \ \lim_{t\to\infty} \|u_\lambda(\cdot,t)\|_\infty = 0\}.$$

Then $0 < \lambda_* < +\infty$. u_λ blows-up in finite time for $\lambda > \lambda_*$.

• $u_{\lambda_*}(x,t)$ is a well-defined L^1-weak solution of $(\mathrm{Q})_p$ (Ni, Sacks, Tzavantis, 1984).

• This solution is global, classical and bounded if $p < p_S$ (Cazenave-Lions, 1984).

• $p \geq p_S$, u_{λ_*} may blow-up in finite time. (for its L^∞-spacial norm)

Two kinds of finite time blow-up are known:

- **Type I blow-up:**

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty < +\infty,$$

the only blow-up possible when $1 < p < p_S$ and $\mathcal{D}$ is convex (Giga and Kohn, 1985, Merle, Zaag, Velázquez, 90s,...).

- **Type II blow-up:**

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty = +\infty.$$

Examples of type II blow-up positive radial solutions: $n \geq 11$ and p large. (Herrero and Velázquez, 1995).

• Radially symmetric blow-up of types I and II: there exists $p_* > p_S$ for which there exists type II blow-up for $p > p_*$, and only type I when $p_S < p < p_*$ (Matano and Merle, 2006, 2011).

• A radial positive type II blow-up solution for $p = p_S$ and $n = 4$ exists (Schweyer, 2013.)

• Is it possible that a classical, positive global solution has infinite-time blow-up?

$$\lim_{t \to +\infty} \|u(\cdot, t)\|_\infty = +\infty$$

Answer: $\mathcal{D} = B(0, 1)$, u radial.

NO if $p < p_S$ (Cazenave and Lions 1984),
NO if $p > p_S$ (Mizoguchi 2005, Quittner and Souplet, 2007)
YES if $p = p_S$ (Galaktionov and Vázquez, 1997; Galaktionov and King, 2003)

Galaktionov and King, 2003: The Ni-Sacks-Tzavantis solution u_{λ^*} is smooth at all times **in the radial case**, and it blows-up in infinite time.

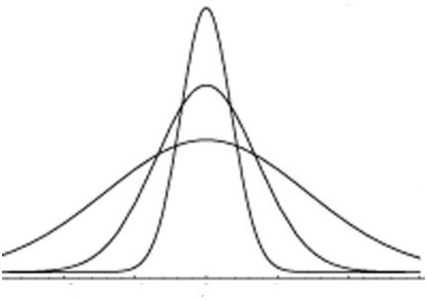
$$u_{\lambda_*}(r,t) \approx \alpha_n \left(\frac{\mu(t)}{\mu(t)^2 + r^2} \right)^{\frac{n-2}{2}}, \quad r = |x|,$$

$0 < \mu(t) \to 0$ as $t \to +\infty$ with a precise estimate for $\mu(t)$.

What about the nonradial case?

We look for a solution

$$u(x,t) \sim \alpha_n \left(\frac{\mu(t)}{\mu^2(t) + |x - \xi(t)|^2} \right)^{\frac{n-2}{2}}$$



Obata (72), Aubin (76), Talenti (76), Gidas-Ni-Nirenberg (81), Caffarelli-Gidas-Spruck (89)

Theorem (C. Cortázar, M.del Pino, M. Musso, 2015)

Assume $n \geq 3$ *and* $q \in \Omega$. *Then there exist functions,*

$$\xi(t) \to q, \quad 0 < \mu(t) \to 0, \quad as \quad t \to +\infty,$$

and a solution $u_q(x,t)$ *of* (P) *of the form*

$$u_q(x,t) = \alpha_n \left(\frac{\mu(t)}{\mu(t)^2 + |x - \xi(t)|^2} \right)^{\frac{n-2}{2}} + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$ *and*

$$\mu(t) \equiv \begin{cases} bt^{-\frac{1}{n-4}} & if \quad n \geq 5 \\ bt^{\frac{1}{4}} e^{-\gamma t^{\frac{1}{2}}} & if \quad n = 4 \\ be^{-\gamma t} & if \quad n = 3 \end{cases}$$

$$\text{(P)}\quad \frac{\partial u}{\partial t} = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0, \infty), \quad u = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

This blow up phenomena is codimension 1 *stable: the initial conditions for Problem (P) near* $u_q(x, 0)$ *leading to infinite time bubbling constitute a* 1 *codimensional manifold.*

If $n \geq 4$,

$$u_q(x,t) \sim \alpha_n \left(\frac{\mu}{\mu^2 + |x-q|^2} \right)^{\frac{n-2}{2}}$$

At main order, away from $x = q$,

$$u_q(x,t) \sim \alpha_n \frac{\mu^{\frac{n-2}{2}}}{|x-q|^{n-2}} - \mu^{\frac{n-2}{2}} H(x,q)$$

with H harmonic in x in Ω, to satisfy Dirichlet boundary condition. Thus

$$u_q(x,t) \sim \mu^{\frac{n-2}{2}} G(x,q)$$

Green's function $G(x,y)$

$$-\Delta_x G(x,y) = c_n \delta(x-y) \quad \text{in } \mathcal{D}, \qquad G(x,y) = 0, \quad x \in \partial\mathcal{D}.$$

$H(x,y)$ the regular part of $G(x,y)$ namely the solution of the problem

$$-\Delta_x H(x,y) = 0 \quad \text{in } \mathcal{D}, \qquad H(x,y) = \Gamma(x-y) \quad \text{for all} \quad x \in \partial\mathcal{D}.$$

$$G(x,y) = \Gamma(x-y) - H(x,y).$$

where Γ is the fundamental solution

$$\Gamma(x) = \frac{\alpha_n}{|x|^{n-2}},$$

Dimension $n = 3$. Strong connection with the Brezis-Nirenberg problem (1983)

$$\Delta u + \lambda u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

There exists a number $\lambda_* > 0$, the Brezis-Nirenberg number, such that a least energy positive solution exists whenever
$\lambda_* < \lambda < \lambda_1(\Omega)$.
When Ω is a ball $\lambda_* = \frac{\lambda_1}{4}$.

In general (conjecture by Brezis-Peletier, proof by Druet 2003):
For $0 < \lambda < \lambda_1$

$$\Delta_x G_\lambda(x,y) + \lambda G_\lambda(x,y) + \gamma \delta_y(x) = 0 \quad \text{in } \Omega, \quad G(\cdot, y) = 0 \quad \text{on } \partial\Omega$$

$$G_\lambda(x,y) = \frac{\alpha_3}{|x-y|} - H_\lambda(x,y).$$

Then

$$\lambda_* = \sup \left\{ 0 < \lambda < \lambda_1 \ / \ \min_\Omega H_\lambda(\xi,\xi) > 0 \right\}.$$

Our result: for a given point $q \in \Omega$, let

$$\lambda_*(q) = \sup \{0 < \lambda < \lambda_1 \ / \ H_\lambda(q,q) > 0\}$$

(so that $H_{\lambda_*}(q,q) = 0$). Then there exists a bubbling solution $u(x,t)$ of (P) with the approximate profile

$$u(x,t) \sim U_{\mu(t)}(x-q) - \mu(t)^{\frac{1}{2}} H_{\lambda_*}(x,q)$$

or, away from q,

$$u(x,t) \sim \mu(t)^{\frac{1}{2}} G_{\lambda^*}(x,q), \quad \mu(t) \sim e^{-2\lambda_* t}$$

If $\Omega = B_1(0)$, $q = 0$, we have

$$\lambda_* = \frac{\pi^2}{4}$$

Simultaneous bubbling at several points?

Let $q_1, \dots, q_k$ be given distinct points in $\mathcal{D}$.

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1,q_1) & -G(q_1,q_2) & \cdots & -G(q_1,q_k) \\ -G(q_1,q_2) & H(q_2,q_2) & -G(q_2,q_3)\cdots & -G(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1,q_k) & \cdots & -G(q_{k-1},q_k) & H(q_k,q_k) \end{bmatrix}$$

Our main result:
A global solution of (P) with its k bubbling points q_j exists if the matrix $\mathcal{G}(q)$ is positive definite.

We can always find k points where $\mathcal{G}(q)$ is positive definite thanks to: $H(x,x) \to +\infty \quad \text{as } \mathrm{dist}\,(x, \partial\Omega) \to 0.$

Assume $\mathcal{G}(q_1, \ldots, q_k)$ is positive definite. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_k) \in (\mathbb{R}_+)^k$

$$I_q(\Lambda) := \frac{1}{2}\sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i<j} G(q_i, q_j)\Lambda_i\Lambda_j - \bar{\alpha}\sum_{j=1}^{k} \Lambda_i^{\frac{4}{n-2}}.$$

If $n \geq 5$, the functional I_q is strictly convex and it has a unique minimizer $\bar{\Lambda}(q)$ which is nondegenerate, namely $D^2_\Lambda I_q(\bar{\Lambda})$ is non-singular.

Theorem (C. Cortázar, M.del Pino, M. Musso, 2015)

Assume $n \geq 5$, $\mathcal{G}(q_1, \cdots, q_k)$ *is positive definite. Then there exist functions,* $j = 1, \ldots, k$,

$$\xi_j(t) \to q_j, \quad 0 < \mu_j(t) \to 0, \quad \text{as} \quad t \to +\infty,$$

and a solution of (P) *of the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}} + \theta(t,x),$$

where $\|\theta(\cdot, t)\|_\infty \to 0$ *and*

$$\mu_j(t) \equiv \beta_n \Lambda_j^{\frac{2}{n-4}} t^{-\frac{1}{n-4}}$$

Here $(\Lambda_1(q), \ldots, \Lambda_k(q))$ *is the unique critical point of* $I_q(\Lambda)$

- The solution found satisfies, for any small δ,

$$\|u(\cdot,t)\|_{L^\infty(\mathcal{D}\setminus\bigcup_{j=1}^k B(q_j,\delta))} \equiv t^{-\frac{n-2}{2(n-4)}}$$

$$\|u(\cdot,t)\|_{L^\infty(B(q_j,\delta))} \sim t^{\frac{n-2}{2(n-4)}}.$$

as $t \to +\infty$.

Assume $n = 4$ and $\mathcal{G}(q_1, \cdots, q_k)$ is positive definite. Then there exists a solution of (P) of the form

$$u(x,t) \sim \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}$$

where $\xi_j(t) \to q_j$ and

$$\mu_j(t) \equiv \beta_4 \Lambda_j t^{-\frac{1}{4}}\, e^{-\beta_4 t^{\frac{1}{2}}}$$

Here $(\Lambda_1(q), \ldots, \Lambda_k(q))$ is the unique unitary eigenvector with all components positive associated to the first eigenvalue of $\mathcal{G}(q_1, \cdots, q_k)$.

Assume $n = 3$ and $\mathcal{G}(q_1, \cdots, q_k)$ is positive definite.

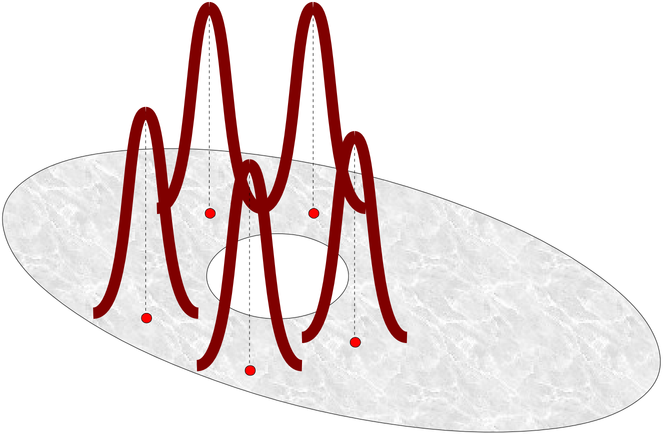
$$\mathcal{G}_\lambda(q) = \begin{bmatrix} H_\lambda(q_1, q_1) & -G_\gamma(q_1, q_2) & \cdots & -G_\lambda(q_1, q_k) \\ -G_\lambda(q_1, q_2) & H_\lambda(q_2, q_2) & -G_\lambda(q_2, q_3) \cdots & -G_\lambda(q_3, q_k) \\ \vdots & & \ddots & \vdots \\ -G_\lambda(q_1, q_k) & \cdots & -G_\lambda(q_{k-1}, q_k) & H_\lambda(q_k, q_k) \end{bmatrix}$$

Let

$$\lambda_*(q) = \sup \{0 < \lambda < \lambda_1 \,/\, \mathcal{G}_\lambda(q) \quad \text{is positive definite}\}.$$

Then there exists a bubbling solution $u(x, t)$ to problem (P) which away from the q_j looks like

$$u(x, t) \approx \sum_{j=1}^{k} \mu_j(t)^{\frac{1}{2}} G_{\lambda_*}(x, q_j), \quad \mu(t) \sim \Lambda_j \, e^{-2\gamma_* t}$$



$$u(x,t) \sim \sum_{j=1}^{5} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}$$

Parallel between this phenomenon problem and existence of of **bubbling solutions** u_ε as $0 < \varepsilon \to 0$ of the Brezis-Nirenberg problem

$$\Delta u + \varepsilon u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathcal{D}, \quad u = 0 \quad \text{on } \partial\mathcal{D}$$

Han (1990), Rey (1991), Bahri-Y.Y. Li-Rey (1995)
Musso-Pistoia (2003): there exists a solution of the form

$$u_\varepsilon(x) \approx \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x - \xi_j^\varepsilon|^2} \right)^{\frac{n-2}{2}}$$

with $\xi_j^\varepsilon \to \bar{q}_j, \quad \mu_{j\varepsilon} \sim (\varepsilon\bar{\Lambda})^{\frac{2}{n-2}}$, and $(\bar{q}, \bar{\Lambda})$ critical point of

$$(q, \Lambda) \mapsto \frac{1}{2} \sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i<j} G(q_i, q_j)\Lambda_i\Lambda_j - \bar{\alpha} \sum_{j=1}^{k} \Lambda_i^{\frac{4}{n-2}}.$$

$$(q,\Lambda) \mapsto \frac{1}{2}\sum_{j=1}^{k} H(q_j,q_j)\Lambda_j^2 - \sum_{i<j} G(q_i,q_j)\Lambda_i\Lambda_j - \bar{\alpha}\sum_{j=1}^{k}\Lambda_i^{\frac{4}{n-2}}.$$

This condition is **much more restrictive** since criticality in the two variables is required. For $k=2$ this reduces to $(\bar{q}_1,\bar{q}_2)$ be a critical point with positive value of the functional

$$(q_1,q_2) \mapsto H(q_1,q_1)^{\frac{1}{2}} H(q_2,q_2)^{\frac{1}{2}} - G(q_1,q_2).$$

No such a pair exists for instance if $\mathcal{D}$ is a ball.

Scheme of the proof.

Fix k points $q_1, \dots, q_k \in \Omega$. Fix $j = 1, \dots, k$.

$$u_j(x,t) = \mu_j(t)^{-\frac{n-2}{2}} U\left(\frac{x-\xi_j(t)}{\mu_j(t)}\right) - h_j(x,t)$$

$$0 < \mu_j(t) \to 0, \quad \xi_j(t) \to q_j, \quad \text{as} \quad t \to +\infty.$$

For each $t > 0$,

$$\Delta_x h_j(x,t) = 0 \quad \text{in } \Omega, \qquad h_j = \mu_j^{-\frac{n-2}{2}} U_{\mu_j,\xi_j} \sim \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x-\xi_j|^{n-2}} \quad \text{on } \partial\Omega\,.$$

Then

$$h_j(x,t) \approx \mu_j(t)^{\frac{n-2}{2}} H(x,\xi_j(t)).$$

Approximation:

$$u^0(x,t) = \sum_{j=1}^{k} u_j(x,t),$$

The error of approximation:

$$\mathcal{E}_0(x,t) := \Delta u^0 + (u^0)^p - u^0_t.$$

Fix $0 < \delta \leq \frac{1}{2} \min_{i \neq j} |q_i - q_j|$ and assume

$$\xi_j(t) \in B(q_j, \delta) \quad \text{for all} \quad t > 0,\ j = 1, \dots, k.$$

We write

$$\mu_j(t) = \mu_0(t)\lambda_j(t), \quad j = 1, \ldots, k,$$

$$\xi_j(t) = \mu_0(t)\zeta_j(t), \quad j = 1, \ldots, k$$

We make the change of variables

$$u(x,t) = \mu_0^{-\frac{n-2}{2}}(\tau) v\left(\frac{x}{\mu_0(\tau)}, \tau\right),$$

where we let $\tau = \tau(t)$ be defined as

$$\frac{d\tau}{dt} = \mu_0^{-2}$$

Then

$$\mu_0^{\frac{n+2}{2}}(\Delta_x u + u^p - u_t) = \Delta_y v + v^p - v_\tau + B[v],$$

with

$$B[v] = \mu_0^{-1}\frac{d\mu_0}{d\tau}\left[\frac{n-2}{2}v + \nabla v \cdot y\right].$$

We call

$$v^0(y,\tau) = \mu_0^{\frac{n-2}{2}}(\tau)u^0(x,t), \quad y = \frac{x}{\mu_0(\tau)}$$

and

$$E(y,\tau) = \Delta_y v^0 + (v^0)^p - \left(\frac{\mu_j}{\mu_0}\right)^2 v_\tau^0 + B[v^0]$$

in

$$\Omega_\tau := \frac{\Omega}{\mu_0(\tau)}.$$

Linearized operator of the equation $\Delta u + u^p = 0$ about the standard bubble U, $p = \frac{n+2}{n-2}$

$$L_0(\phi) := \Delta\phi + pU^{p-1}\phi.$$

All bounded solutions of $L_0(\phi) = 0$ are linear combinations of

$$\begin{aligned} Z_j(y) &:= \left.\frac{\partial U_{\mu,\xi}}{\partial \xi_j}\right|_{\xi=0,\mu=1} = \frac{\partial U}{\partial y_j}(y), \quad j = 1, \dots, n, \\ Z_{n+1}(y) &:= \left.\frac{\partial U_{\mu,\xi}}{\partial \mu}\right|_{\xi=0,\mu=1} = \frac{n-2}{2}U(y) + y \cdot \nabla U(y). \end{aligned}$$

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N)$$

has exactly one negative eigenvalue λ_0, with a positive, radial eigenfunction $Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\lambda_0|}\,|y|}$.

We want the $L^2(dy)$ projections of the error $E(y,\tau)$ onto $Z_1(y), \dots Z_{n+1}(y)$ to be small

$$(I) \qquad \int_{|y-\zeta_j|<\frac{\delta}{\mu_0\lambda_j}} EZ_{n+1}\left(\frac{y-\zeta_j}{\lambda_j}\right) dy \approx 0,$$

and

$$(II) \qquad \int_{|y-\zeta_j|\le\frac{\delta}{\mu_0\lambda_j}} EZ_l\left(\frac{y-\zeta_j}{\lambda_j}\right) dy \approx 0, \quad l = 1, \dots, n$$

for any $j = 1, \dots, k$.

From (I):

$$\mu_0^{n-2}\left[H(q_j,q_j)\lambda_j^{n-3}-\sum_{i\neq j}\lambda_j^{\frac{n-2}{2}-1}\lambda_i^{\frac{n-2}{2}}G(q_j,q_i)\right]+c_0\lambda_j\mu_0^{-1}\frac{d\mu_0}{d\tau}\ \sim\ 0.$$

Choose

$$\mu_0^{-1}\frac{d\mu_0}{d\tau}=-\frac{\mu_0^{n-2}}{n-2},\quad \mu_0(\tau)=\tau^{-\frac{1}{n-2}}$$

and $\lambda_j(\tau):=b_j+o(1)$, with

$$H(q_j,q_j)b_j^{n-3}-\sum_{i\neq j}b_j^{\frac{n-2}{2}-1}b_i^{\frac{n-2}{2}}G(q_j,q_i)=\frac{c_0}{n-2}b_j.$$

Relations

$$H(q_j,q_j)b_j^{n-3}-\sum_{i\neq j} b_j^{\frac{n-2}{2}-1}b_i^{\frac{n-2}{2}}G(q_j,q_i)=\frac{c_0}{n-2}b_j.$$

are equivalent to being a critical point of the functional

$$I_q(b):=\frac{1}{2}\sum_{j=1}^{k}H(q_j,q_j)b_j^{n-2}-\sum_{i<j}G(q_i,q_j)(b_ib_j)^{\frac{n-2}{2}}-\bar{\alpha}\sum_{j=1}^{k}b_i^2.$$

This functional is convex thanks to the hypothesis and it has a unique minimizer $(b_1,\dots,b_k)$ which we fix in what follows.

From (II):

$$\frac{d\xi_j}{d\tau} = -\mu_0^n B_j,$$

$$B_j = \left[b_j^n \nabla_x H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n+2}{2}} b_i^{\frac{n-2}{2}} \nabla_x G(q_j, q_i) \right].$$

Choose

$$\xi_j(\tau) = q_j - \mu_0^2 B_j (1 + o(1))$$

We specify that

$$\mu_j(t) \,=\, \mu_0(t)\,[b_j + \mu_{1,j}(t)]\,, \quad j = 1, \ldots, k,$$

$$\xi_j(t) = q_j + \mu_0^2(t)\,[B_j + \xi_{1,j}(t)]\,, \quad j = 1, \ldots, k,$$

for a certain function μ_0 and positive constants b_j, B_j already found, with smaller corrections $\mu_{1,j},\ \xi_{1,j}$ to be found.

Complete ansatz:

$$v(y,t) = v_0(y,\tau) + \sum_{j=1}^{k} e_j(\tau) Z_0\left(\frac{y-\zeta_j}{\lambda_j}\right) + \phi(y,\tau)$$

We decompose

$$\phi = \sum_{j=1}^{k} \eta_j \phi_j + \psi, \quad \text{with} \quad \eta_j(y) = \eta\left(\frac{|y-\zeta_j|}{R}\right)$$

where ψ solves a problem of the form

$$\psi_\tau = \Delta\psi + V\psi - B[\psi] + f(y,\tau) \quad \text{in} \quad \Omega_\tau$$

$$\psi(\cdot,\tau) = 0, \quad \text{on} \quad \partial\Omega_\tau, \quad \psi(y,\tau_0) = g(y), \quad \text{on} \quad \Omega_{\tau_0}$$

where

$$V = p(v^0)^{p-1}\left(1 - \sum_{j=1}^{k} \eta_j\right)$$

The idea is to solve for ϕ_j the modified problem in $B(\zeta_j, R)$

$$(\phi_j)_\tau = \Delta\phi_j + p v_0^{p-1}\phi + B[\phi_j] + E_1(y,\tau) + N(\phi) + \sum_{l=0}^{n+1} c_l(\tau) Z_l(\frac{y-\zeta_j}{\lambda_j})$$

$$\int \phi_j(y,\tau) Z_l(\frac{y-\zeta_j}{\lambda_j})\, dy = 0 \quad \text{for all} \quad l = 0, 1, 2, \dots, n+1.$$

where E_1 is the new error of approximation.
After solving this we adjust the parameters $\mu_{1,j}, \xi_{1,j}$ and e to get $c_l \equiv 0$. These quantities satisfy a weakly coupled system of ODEs

Each ϕ_j solves in $B(\zeta_j, R)$ a problem of the form

$$\phi_\tau = \Delta\phi + pU^{p-1}\phi + h(y,\tau) - \sum_{l=0}^{n+1} c_l(\tau) Z_l, \quad y \in B(0,R),$$

$$\phi(\cdot, \tau_0) = 0, \quad \int_{B(0,R)} \phi\, Z_l\, dy = 0 \quad \text{for all} \quad l = 0,1,2,\ldots,n+1.$$

Here

$$c_l(\tau) = \int_{B(0,R)} h\, Z_l\, dy,$$

We prove that: if

$$|h(y,\tau)| \le C \frac{\tau^{-\nu}}{(1+|y|^{n+a})}, \quad \text{for some} \quad 2 < a < 3$$

then there exists a solution ϕ with

$$|\phi(y,\tau)| \le C\, R^n \, \frac{\tau^{-\nu}}{(1+|y|^{n+a-2})}.$$

The basic fact is a linear energy bound: for all $R \gg 1$ we have

$$\int_{B(0,R)} |\nabla \phi|^2 - pU^{p-1}\phi^2 \ge \frac{c}{R^2} \int_{B(0,R)} \phi^2$$

for all $\phi \in H_0^1(B(0,R))$ with

$$\int_{B(0,R)} \phi Z_l = 0 \quad \text{for all} \quad l = 0, 1, 2, \ldots, n+1.$$

Now we solve

$$c_l(\tau) \simeq 0$$

which reduces to solving a system for the functions $\mu_{1,j}, \xi_{ij}, e_j$:

$$\int_{|y|\le \mu_0^{-1}} E_1 Z_l \, dy \approx 0 \quad \text{for all} \quad l, j.$$

They become

$$\frac{d\mu_{1,j}}{d\tau} \; + \; \frac{c}{\tau}\sum_{i=1}^{k} M_{ij}\,\mu_{1,i} = O(\mu_0^{n-1+\sigma})$$

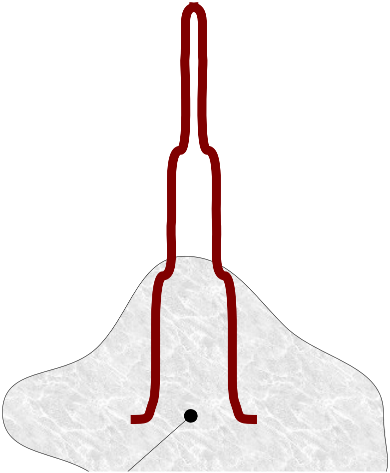
$$\frac{d\xi_{1,j}}{d\tau} \quad = O(\mu_0^{n+\sigma})$$

$$\frac{de_j}{d\tau} - |\lambda_0| e_j = O(\mu_0^{n-2+\sigma})$$

Here $M = [M_{ij}] = D^2 J_q(b)$, where b is the unique minimizer of

$$J_q(a_1,\dots,a_k) := \frac{1}{2}\sum_{j=1}^{k} a_j^{n-2} - \sum_{i<j} G(q_i,q_j)(a_i a_j)^{\frac{n-2}{2}} - \bar{\alpha}\sum_{j=1}^{k} a_j^2.$$

Multiple bubbling at a single point?



For the Brezis-Nirenbrg problem

$$\Delta u + \varepsilon u + u^{\frac{n+2}{n-2}} = 0, \quad u > 0 \quad \text{in } \mathcal{D}, \quad u = 0 \text{ on } \partial\mathcal{D}$$

NO for positive solutions Schoen, Y.Y. Li:

YES for sign-changing solutions Iacopetti-Vaira 2014
In dimension $n \geq 7$ and $\mathcal{D}$ the unit ball, has solutions with multiple bubbling at the origin when $0 < \varepsilon \to 0$:

$$u_\varepsilon(x) \approx \sum_{j=1}^{2} (-1)^j \alpha_n \left(\frac{\mu_j^\varepsilon}{(\mu_j^\varepsilon)^2 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \mu_2^\varepsilon \ll \mu_1^\varepsilon.$$

The analogue of this in the parabolic setting?

Theorem (M.del Pino, M. Musso, J.Wei 2015)

Assume $n \geq 7$ and $q \in \Omega$. Then there exist functions, $j = 1, 2$,

$$\xi_j(t) \to q, \quad 0 < \mu_j(t) \to 0, \quad as \quad t \to +\infty,$$

and a solution of (P) *of the form*

$$u(x,t) = \sum_{j=1}^{2} (-1)^j \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}} + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$, as $t \to \infty$, and

$$\mu_1(t) \simeq t^{-\frac{1}{n-4}}, \quad \mu_2(t) \simeq t^{-\frac{3n-10}{(n-4)(n-6)}}$$

THANKS