On Serrin's over-determined problem and a conjecture of Berestycki, Caffarelli and Nirenberg

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Joint work with Manuel del Pino and Frank Pacard, Kelei Wang

Method of Moving Planes

In a classical paper, Gidas-Ni-Nirenberg 1979 introduced the method of moving planes to prove the following classical result:

Let $\Omega = B_R(0)$ be a ball and u be a solution of

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{0.1}$$

Then u must be radially symmetric.

Serrin's Over-determined Problem

In fact the method of moving planes was used by Serrin in 1971. He considered the following overdetermined problem: Let Ω be a bounded domain and u be a solution of

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{Constant on } \partial\Omega \end{cases} \tag{0.2}$$

Then Ω must be a ball.

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Then Ω must be a ball.

(Serrin first considered the case $f(u) = 1$. Then at the end of the paper he extended to general $f(u)$.)

Serrin's Overdetermined Problem

We reformulate Serrin's problem: Find a domain Ω such that there exists a solution u to

$$(S) \quad \begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{Constant on } \partial\Omega \end{cases}$$

Serrin's result: The only bounded domain is ball.

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A compact, connected, embedded hyper-surface in R^N whose mean curvature is constant, must necessarily be an Euclidean sphere.

Serrin had a clever insight into the geometric structure of Problem (S) to prove his result as an analog of Alexandrov's. His insight is philosophical.

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The purpose of this talk is to further explore the parallel between Alexandrov's and Serrin's statements. The underlying question is:

Is there a realistic link between embedded constant mean curvature (CMC) surfaces and domains where Serrin's problem (S) is solvable?

Serrin's Problem in Unbounded Domains

In this talk, we consider Serrin's problem when Ω becomes unbounded.

When $f(u) = 0$,

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{Constant on } \partial\Omega \end{cases}$$

This kind of domains are called exceptional domains and the function u is called root function. In $\mathbb{R}^2$, an example of exceptional domain is

$$\Omega = \{x \in \mathbb{R}^2 \mid |x_2| < \frac{\pi}{2} + \cosh(x_1)\}$$

Traizet, GAFA 2014 gave a complete classification of two-dimensional exceptional domains.

Helen, Hauswirth, Pacard, PJM 2011 derived Weierstrass representation for such domains in $\mathbb{R}^3$.

In this talk, we consider $f \neq 0$ is nonlinear and Ω is unbounded.

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For the purpose of the talk, we may take

$$f(u) = u - u^3 \quad \text{(Allen-Cahn type nonlinearity)}$$

All the results of this talk are true as long as the following holds: there exists a unique ODE solution

$$w^{''} + f(w) = 0, w(0) = 0, w(+\infty) = C$$

Examples of

$$f(u) = u - u^2 (KPP)$$

$$f(u) = u - u^3 (Allen - Cahn), ...$$

Berestycki-Caffarelli-Nirenberg Conjecture

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• the complement of a ball or cylinder

B-C-N Conjecture for epigraphs

We discuss first the B-C-N Conjecture in the case of epigraphs:

$$\Omega = \{x_N > \varphi(x^{'})\}$$

where $x^{'} = (x_1, ..., x_{N-1})$ and φ is Locally Lipstchitz. The original paper of Berestycki-Caffarelli-Nirenberg mainly considered this case.

With this notation, Serrin's problem becomes

$$(S) \quad \begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega = \{x_N > \varphi(x^{'})\} \\ u > 0 \quad \text{in } \{x_N > \varphi(x^{'})\}, \\ u = 0 \quad \text{on } \partial\Omega = \{x_N = \varphi(x^{'})\}, \\ \frac{\partial u}{\partial \nu} = C \quad \text{on } \partial\Omega = \{x_N = \varphi(x^{'})\} \end{cases}$$

An Obvious Solution: $\varphi = 0, u = w(x_N)$ which satisfies an ODE

$$w^{''} + f(w) = 0, w(0) = 0, w^{'}(0) = C$$

$$w(x_N) = \tanh(\frac{x_N}{\sqrt{2}})$$

Another Obvious Solution: $\varphi = ax^{'} + b$, $u = w(\frac{x_N - ax^{'} - b}{\sqrt{1+|a|^2}})$

The question is then: are these the only solutions?

Berestycki-Cafferalli-Nirenberg Conjecture for epigraph:

The epigraph $\{x_N > \varphi(x^{'})\}$ must be a half space, i.e.

$$\varphi(x^{'}) = a \cdot x^{'} + b$$

$$u = w(\frac{x_N - ax^{'} - b}{\sqrt{1 + |a|^2}})$$

Previous Results

Yes, if φ satisfies

$$\lim_{|x^{'}|\to+\infty}[\varphi(x^{'}+\tau)-\varphi(x^{'})]=0, \forall\tau\in\mathbb{R}^{N-1}$$

Berestycki, Cafferalli, Nirenberg (CPAM 1997)

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Yes, if φ is globally Lipschitz, and $N = 2, 3$

Farina, Valdinoci (2010)

Positive Answers to BCN Conjecture

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$$\lim_{|x'| \to +\infty} \varphi(x') = +\infty$$

- $3 \leq N \leq 8$, provided that u satisfies a natural monotonicity condition

$$\frac{\partial u}{\partial x_N} > 0$$

Remark on the natural condition

When $3 \leq N \leq 8$, we require an additional assumption that

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Theorem 2 (del Pino, Pacard, Wei arxiv 2014, Duke Math. J.) In Dimension $N \geq 9$ there exists a solution to Problem (S) in an entire epigraph Ω which is not a half-space.

We now compare B-C-N Conjecture for epigraphs with another conjecture:

De Giorgi's conjecture (1978): *Let u be a bounded solution of equation*

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N, \tag{AC}$$

which is monotone in one direction, say $\partial_{x_N} u > 0$. Then, **at least** *when $N \leq 8$, there exist p, ν such that*

$$u(x) = \tanh\left(\frac{(x-p)\cdot\nu}{\sqrt{2}}\right).$$

Progress on De Giorgi's Conjecture

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- True for $4 \leq N \leq 8$ Savin (2009), if in addition

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}.$$

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- False for $N \ge 9$ del Pino-Kowalczyk-Wei (2011)

B-C-N Conjecture on epigraphs

- True for $N = 2$, Wang-Wei 2015
- True for Lipschitz graph or coercive graph, Wang-Wei 2015
- True for $3 \leq N \leq 8$, Wang-Wei 2015, provided that

$$\frac{\partial u}{\partial x_N} > 0$$

- False for $N \geq 9$, del Pino-Pacard-Wei 2014

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What about $N \geq 3$? Surprisingly it is False in dimension $N \geq 3$!!!

We first describe the epigraph of Theorem 2. We start with the famous

The minimal surface equation

$$H_\Gamma := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \Omega \subset \mathbb{R}^{N-1}.$$

$$\Gamma = \{(x', F(x')) \in \mathbb{R}^{N-1} \times \mathbb{R} \ / \ x' \in \Omega \subset \mathbb{R}^{N-1}\}$$

is a minimal surface (minimal graph) in $\mathbb{R}^N$

Problem (Bernstein, 1910): Are all (entire) solutions of the minimal surface equation

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Yes, when $N \leq 8$: Bernstein (1910), Fleming (1962) $N = 3$;De Giorgi (1965) $N = 4$;Almgren (1966), $N = 5$; Simons (1968), $N = 6, 7, 8$.

False for $N \geq 9$:

• Bombieri-De Giorgi-Giusti found a counterexample (1969).



The BDG minimal graph:

Explicit construction by super and sub-solutions, $N = 9$, of a non-trivial solution of

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

$$F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto F(|\mathbf{u}|, |\mathbf{v}|).$$

In addition, $F(|\mathbf{u}|, |\mathbf{v}|) > 0$ for $|\mathbf{v}| > |\mathbf{u}|$ and

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|).$$

Polar coordinates:

$$|\mathbf{u}| = r\cos\theta,\ \ |\mathbf{v}| = r\sin\theta, \quad \theta \in (0, \frac{\pi}{2})$$

We have that for large r,

$$F(r,\theta) \approx F_0(r,\theta) = r^3 g(\theta)$$

$$g > 0 \text{ in } (\frac{\pi}{4}, \frac{\pi}{2}], \quad g(\frac{\pi}{2} - \theta) = -g(\theta), \quad g'(\frac{\pi}{2}) = 0.$$

$$g(\theta) \sim \cos(2\theta)$$

and g is such that

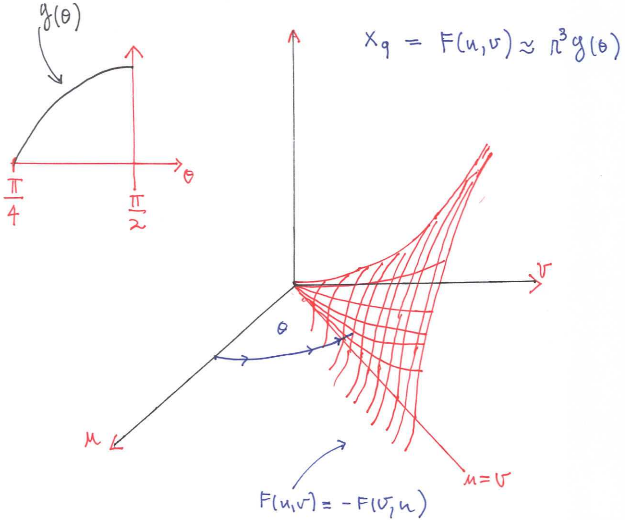
$$\nabla \cdot \left(\frac{\nabla F_0}{|\nabla F_0|} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

Equivalent to an ODE for g

$$\frac{21g\,\sin^3 2\theta}{\sqrt{9g^2+{g'}^2}} + \left(\frac{g'\sin^3 2\theta}{\sqrt{9g^2+{g'}^2}}\right)' = 0 \quad \text{in} \ \left(\frac{\pi}{4},\frac{\pi}{2}\right),$$

$$g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right).$$

This problem has a solution $g > 0$ in $(\frac{\pi}{4},\frac{\pi}{2}]$.



g(θ)
π/4
π/2
θ
X_q = F(u,v) ≃ r³g(θ)
v
u
θ
u=v
F(u,v) = -F(v,u)

Asymptotic behavior of F

Asymptotic behavior of BDG surface $x_9 = F(r, \theta)$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$:
$\sigma \in (0, 1)$

$$F(r, \theta) = r^3 g(\theta) + O(r^{-\sigma}) \quad \text{as } r \to +\infty.$$

(del Pino, Kowalczyk, Wei, 2011)

$$F(r, \theta) = r^3 g(\theta) + O(r^{-1}) \quad \text{as } r \to +\infty.$$

(Daskalopoulous, del Pino, Davila, Wei, 2014)

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What is the epigraph? Let $\Omega_{bdg} = \{x_9 > F(x^{'})\}$ where F is Bombieri-De Giorgi-Giusti minimal graph. Then, for a sufficiently small $\epsilon > 0$, the epigraph Ω in Theorem 2 lies in a $O(\epsilon)$-neighborhood of $\epsilon^{-1}\Omega_{bdg}$.

The principle behind Theorem 2 applies, more generally, to domains enclosed by a large dilation of an embedded minimal surface or CMC surface, provided that sufficient information about the surface (such as nondegeneracy) is available.

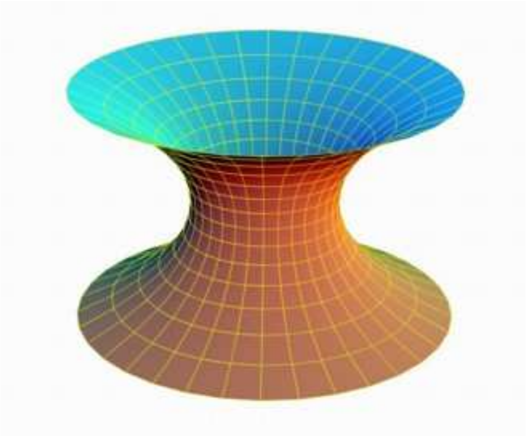
Theorem 3 (del Pino-Pacard-Wei 2014) Let Γ be a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and non-degenerate (i.e. the only bounded kernel of its Jacobi opetaor

$$J[h] = \Delta_\Gamma h + |A|^2 h = 0$$

are rigid motions.) Let Γ separate $\mathbb{R}^3$ into two connected parts Ω_1 and Ω_2. Then for any sufficiently small $\epsilon > 0$ Serrin's problem can be solved in either $\epsilon^{-1}\Omega_1 + O(1)$ or $\epsilon^{-1}\Omega_2 + O(1)$

Examples: nondegeneracy and Morse index are known for the catenoid and Costa, Costa-Hoffmann-Meeks surfaces (Nayatani (1990), Morabito, (2008)).

Catenoid



On the other hand, a statement similar to Theorem 3 holds for more general CMC surfaces, i.e.

$$H \equiv C$$

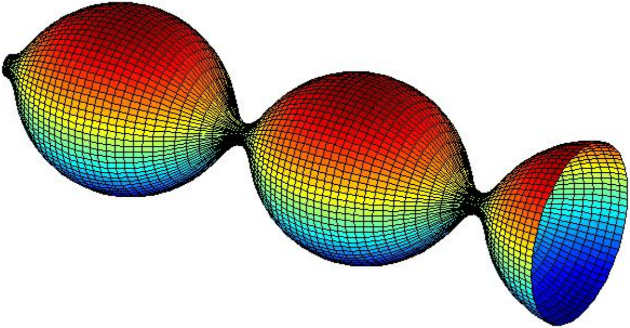
Examples of CMC surfaces

•• balls

•• cylinder

•• Delaunay surfaces D_τ

Delaunay surfaces



We have the validity of the following result.

Theorem 4. Let Ω be the inside of the Delaunay surface. For each $\epsilon > 0$ sufficiently small there exists a domain of revolution Ω, which lies within a ϵ-neighborhood of the region $\epsilon^{-1} D_\tau$, such that Serrin's over-determined Problem (S) is solvable.

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Counter-example to B-C-N Conjecture

The last two Theorems gives a counter-example to the B-C-N Conjecture for dimensions $N \geq 3$.

Berestycki-Caffarelli-Nirenberg Conjecture (1997): is it true that an unbounded domain Ω where problem (S) is solvable must be either

• a half-space (in the case of epigraph), or

• a cylinder $\Omega = B_k \times \mathbb{R}^{N-k}$, where B_k is a k-dimensional Euclidean ball, or

• the complement of a cylinder or ball?

Summary on B-C-N Conjecture

B-C-N Conjecture for epigraph:

•• True for $N = 2$; Conditionally true for $3 \leq N \leq 8$; False for $N \geq 9$

B-C-N Conjecture for general unbounded domains:

•• True for $N = 2$; False for $N \geq 3$

We now discuss the proofs of Theorem 1 (positive answer) and Theorem 2 (negative answer) on Serrin's problem on epigraphs

$$(S) \quad \begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega = \{x_N > \varphi(x')\} \\ u > 0 \quad \text{in } \{x_N > \varphi(x')\}, \\ u = 0 \quad \text{on } \partial\Omega = \{x_N = \varphi(x')\}, \\ \frac{\partial u}{\partial \nu} = C \quad \text{on } \partial\Omega = \{x_N = \varphi(x')\} \end{cases}$$

Proof of Theorem 1

Our first key observation is that the boundary Neumann data (Constant)

$$\frac{\partial u}{\partial \nu} = \text{C} \quad \text{on } \partial\Omega$$

can not be arbitrary. In fact we can prove

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Theorem 5 (Wang-Wei): Let u be a solution of Serrin's over-determined problem on an epigraph. Then

$$\frac{\partial u}{\partial \nu} = -\sqrt{2F(0)} \;\; \text{on} \;\; \partial\Omega$$

where

$$F(u) = \frac{1}{4}(1-u^2)^2$$

This is mainly because $\{u > 0\}$ is an epigraph, we can touch $\partial\{u > 0\}$ by arbitrarily large balls from both sides and compare the solution with the solution in big ball

$$\begin{cases} \Delta v^R + f(v^R) = 0, & \text{in } B_R, \\ v^R > 0, & \text{in } B_R, \\ v^R = 0, & \text{on } \partial B_R. \end{cases}$$

Then we construct suitable comparisons in these balls to determine $|\nabla u| \lfloor_{\partial\Omega}$.

Main idea: since $\frac{\partial u}{\partial \nu} \equiv C$, we obtain lower and upper bounds from both sides.

As a consequence of Theorem 5 a solution to the Serrin's over-determined problem must also satisfy

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \Omega = \{u > 0\}, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ |\nabla u| = \sqrt{2F(0)}, & \text{on } \partial\Omega \end{cases} \tag{0.3}$$

which is the Euler-Lagrange equation for the following one-phase free boundary energy functional

$$\int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\chi_{\{u>0\}}. \tag{0.4}$$

With the monotonicity condition

$$\frac{\partial u}{\partial x_N} > 0$$

we further show that a solution to Serrin's overdetermined problem (S) is necessarily a global minimizer of (**??**).

Hence the proof of Theorem 1 is reduced to the study of solutions to the following one phase free boundary problem:

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \Omega = \{u > 0\}, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ |\nabla u| = \sqrt{2F(0)}, & \text{on } \partial\Omega. \end{cases} \tag{0.5}$$

For this one phase free boundary problem, we have the following estimates

1) (Modica estimate)

$$\frac{1}{2}|\nabla u|^2 \leq \frac{1}{4}(1-u^2)^2$$

2) (Monotonicity Formula)

$$E(r;u,x) := r^{1-N} \int_{B_r(x)\cap\Omega} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2)\chi_{\{u>0\}}$$

is non-decreasing in $r > 0$.

3) (Energy Estimates) For global minimizers there holds

$$E(R;u,x) \leq cR^{N-1}.$$

With these estimates at hand, we are in a similar situation of De Giorgi's Conjecture.

The proof of B-C-N Conjecture in dimension two

The key to proving B-C-N conjecture is to prove the following energy bound without requiring the minimality of the energy:

$$\int_{B_R(0)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}F(u)\chi_{\{u>0\}} \leq C\ R.$$

In $\mathbb{R}^2$, we can achieve this, since by the result of Caffarelli-Cordova (2006), the Modica estimate implies that the set $\{u = 0\}$ is a mean convex set (and hence convex set). Hence the function φ is concave.

For example we can estimate

$$\begin{aligned}
& \int_{B_R(0)\cap\{u>0\}} (f(u)) \\
= & -\int_{B_R(0)\cap\{u>0\}} \Delta u \\
= & -\int_{\partial B_R(0)\cap\{u>0\}} \frac{\partial u}{\partial r} + \int_{B_R(0)\cap\partial\{u>0\}} |\nabla u| \\
\leq & \; CR.
\end{aligned}$$

Since $F(u) \leq Cf(u)$ for $u > \gamma$, this gives

$$\int_{B_R(0)\cap\{u>\gamma\}} F(u) \leq CR$$

On the other hand, by Berestycki-Caffarelli-Nirenberg,

$$\{u \leq \gamma\} \subset \{x : dist(x, \partial\{u > 0\}) \leq M\}$$

Using convexity of the set $\partial\{u > 0\}$, we have

$$|\{x : dist(x, \partial\{u > 0\}) \leq M\} \cap B_R| \leq CR$$

Hence

$$\int_{B_R(0)\cap\{u\leq\gamma\}} F(u) \leq CR$$

$$\int_{B_R(0)\cap\{u>0\}} F(u) \leq CR.$$

By Modica's estimate

$$\int_{B_R(0)\cap\{u>0\}} (|\nabla u|^2 + F(u)) \leq CR.$$

For dimensions $3 \leq N \leq 8$, we use the extra condition

$$\frac{\partial u}{\partial x_N} > 0$$

to show that the solution is a global minimizer, i.e.

$$E_\Omega[u] \leq E_\Omega[v], \quad \forall v = u \text{ on } \partial\Omega$$

From this we derive the following key energy estimates

$$E_{B_R}[u] \leq cR^{N-1}.$$

Then we are in a similar situation as in Savin's proof of De Giorgi's conjecture. We follow the new proof of Kelei Wang (2014)

Summary

1. We proved that the Serrin's problem is a stationary solution of the one-phase free boundary problem

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \Omega = \{u > 0\}, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ |\nabla u| = \sqrt{2F(0)}, & \text{on } \partial\Omega. \end{cases}$$

2. When $3 \leq N \leq 8$ We use monotonicity condition to show that it is a global minimizer. A crucial energy bound is obtained. When $N = 2$ we use Modica's estimate to obtain the energy bound.

3. Then we use a blown-down process to show that $\{u_\epsilon = 0\}$ is a minimal graph.

Proof of Theorem 2

Theorem 2 (del Pino, Pacard, Wei) In Dimension $N \geq 9$ there exists a solution to Problem (S) in an entire epigraph Ω which is not a half-space.

Theorem 2 will be proved by gluing method.

New ingredients of gluing methods

1. The linearized problem is an over-determined Cauchy problem in half space:

$$\Delta\phi + f'(w(t))\phi = \alpha(y)\, w'(t) + g(y,t) \quad \text{in } \mathbb{R}^9_+,$$
$$\phi(y,0) = 0 \qquad \text{for all} \quad y \in \mathbb{R}^8,$$
$$\partial_t\phi(y,0) = \beta(y) \quad \text{for all} \quad y \in \mathbb{R}^8.$$

2. The reduced problem for adjustment of the graph involves slow-decaying sources

$$\Delta_\Gamma h_1 + |A|^2 h_1 = O(\frac{1}{r^2})$$

$$\Delta_\Gamma h_2 + |A|^2 h_2 = O(\frac{1}{r^3})$$

$$\Delta_\Gamma h_3 + |A|^2 h_3 = O(\frac{1}{r^4})$$

We need to develop new theory for the Jacobi operator.

The Laplacian near Γ

Let $\Gamma = \{x_9 = F(x')\}$ be the B-D-G minimal graph.
For a certain $\delta > 0$ we consider the Fermi Coordinate

$$x = X(z,y) := y + z\nu(y), \quad y \in \Gamma, \quad |z| < \delta r(y) \tag{0.6}$$

defines diffeomorphism onto an expanding tubular neighborhood of Γ. Let us consider the manifold

$$\Gamma^z := \{y + z\nu(y) \; / \; y \in \Gamma\}$$

The Euclidean Laplacian in $\mathbb{R}^9$ near Γ can be expressed in these coordinates by the well-known formula

$$\Delta_x = \partial_z^2 + \Delta_{\Gamma^z} - H_{\Gamma^z}(y)\partial_z \tag{0.7}$$

where $H_{\Gamma^z}(y)$ denotes mean curvature of Γ^z at the point $y + z\nu(y)$ and the operator Δ_{Γ^z} is understood to act on functions of the variable y.

On the other hand, is is well-known that if $k_1, \ldots, k_8$ denote the principal curvatures of Γ, then

$$H_{\Gamma^z}(y) \; = \; \sum_{i=1}^{8} \frac{k_i(y)}{1 - zk_i(y)}$$

Since Γ is a minimal surface we have that $\sum_{i=1}^{8} k_i = 0$, therefore

$$H_{\Gamma^z}(y) \; = \; z|A_\Gamma|^2 + z^2 \sum_{i=1}^{8} k_i^3 + z^3 \sum_{i=1}^{8} k_i^4 + z^4 \theta(y, z) \qquad (0.8)$$

where

$$|A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2 := O(r^{-2}), \qquad (0.9)$$

Coordinates near Γ_ϵ

For some $\delta > 0$, the following map defines coordinates for a expanding neighborhood of Γ_ϵ:

$$x = X(y,z) := y + z\nu(\alpha y), \quad y \in \Gamma_\epsilon,\ |z| < \delta\epsilon^{-1}\,\mathtt{r}(\epsilon y) \tag{0.10}$$

is computed as

$$\Delta = \partial_z^2 + \Delta_{\Gamma_\epsilon^{\epsilon z}} - \epsilon H_{\Gamma^{\epsilon z}}(\epsilon y)\,\partial_z \tag{0.11}$$

where now

$$\Delta_{\Gamma_\epsilon^{\epsilon z}} = \Delta_{\Gamma_\epsilon} + \epsilon z a_{ij}^1(\epsilon \mathtt{y}, \epsilon z)\partial_{ij}^2 + \epsilon^2 z b_j^1(\epsilon \mathtt{y}, \epsilon z)\partial_j. \tag{0.12}$$

and

$$\epsilon H_{\Gamma^{\epsilon z}}(\epsilon y) = \epsilon H_\Gamma + \epsilon^2 z|A_\Gamma(\epsilon y)|^2 + \epsilon^3 z^2 \sum_{i=1}^{8} k_i(\epsilon y)^3 +$$

$$\epsilon^4 z^3 \sum_{i=1}^{8} k_i^4(\epsilon y) + \epsilon^5 z^4 \theta(y,z) \tag{0.13}$$

The perturbed epigraph

We consider now a bounded smooth function $h(y)$ defined on Γ and the coordinates near Γ_ϵ,

$$x = X^h(y,t) := y + (t + h(\epsilon y))\nu(\epsilon y), \quad y \in \Gamma_\epsilon,\ |t| < \delta\epsilon^{-1}\,\mathtt{r}(\epsilon y) \tag{0.14}$$

(Following an idea of del Pino-Kowalczyk-Wei, 2011)
We fix a positive number M and assume for the moment that h is a smooth function such that

$$\|D^2_\Gamma h\|_{L^\infty(\Gamma)} \ + \ \|D_\Gamma h\|_{L^\infty(\Gamma)} + \|h\|_{L^\infty(\Gamma)} \ \le \ M. \tag{0.15}$$

Under the assumptions made the set

$$\Gamma^h_\epsilon = \{y + h(\epsilon y)\nu(\epsilon y)\ /\ y \in \Gamma_\epsilon\}$$

turns out to be the graph of a function $F^h_\epsilon(x')$.
Thus we consider the epigraph

$$\Omega^h_\epsilon := \{(x', x_9)\ /\ x_9 > F^h_\epsilon(x')\}$$

whose boundary is Γ^h_ϵ.

The problem and a first approximation

We want to solve the problem

$$S[u] := \Delta u + f(u) = 0 \quad \text{in } \Omega_\epsilon^h$$

$$u = 0, \quad \partial_\nu u = constant \quad \text{on } \Gamma_\epsilon^h$$

We observe that in the coordinates

$$x = y + (t + h(\epsilon y))\, \nu(\epsilon y), \quad y \in \Gamma_\epsilon, \quad |t| < \delta \epsilon^{-1} \mathtt{r}_\epsilon(y).$$

we have that $x \in \Omega_\epsilon^h$ if and only if $t > 0$. The problem for then becomes

$$\begin{aligned} S[u] \;=\; \Delta_x u \;+\; f(u) \;=\; 0 & \quad \text{in } \Omega_\epsilon^h, \\ u(y,0) \;=\; 0 & \quad \text{for all} \quad y \in \Gamma_\epsilon, \\ \partial_t u(y,0) \;=\; constant & \quad \text{for all} \quad y \in \Gamma_\epsilon. \end{aligned} \tag{0.16}$$

We have the existence of a unique solution $w(t)$ to the problem

$$w'' + f(w) = 0 \quad \text{in } (0,\infty),$$

$$w(0) = 0, \quad w(+\infty) = 1.$$

As a first approximation, close to Γ_ϵ^h we then take

$$u_0(x) := w(t).$$

Error

$$
\begin{aligned}
S[u_0] &= \epsilon H w' \\
&-\epsilon^2[\Delta_\Gamma h + |A_\Gamma|^2\, h\,]\, w' \;+\; \epsilon^2 a_{ij}\,\partial_i h\,\partial_j h\, w'' \\
&-\,[\epsilon^2 t|A_\Gamma|^2 w' + \epsilon^3 \sum_{i=1}^{8} k_i^3 (t+h)^2 w' + \epsilon^4 \sum_{i=1}^{8} k_i^4 (t+h)^3 w'] \\
&-\,\{\,\epsilon^3 (t+h)\,[a_{ij}^1\,\partial_{ij} h \;+\; b_j^1\,\partial_j h\,]\, w' \;+\; \epsilon^5\,(t+h)^4 \theta\,\}\, w'
\end{aligned}
\tag{0.17}
$$

where all coefficients are evaluated at $(\epsilon \mathrm{y}, \epsilon(t + h(\epsilon \mathrm{y}))$ or $\epsilon \mathrm{y}$.

What we will do is to improve this first approximation by choosing h in such a way that at main order the relation

$$\int_0^{+\infty} S[u_0]\,(y,t)\,w'(t)\,dt = 0 \quad \text{for all} \quad y \in \Gamma_\epsilon$$

is satisfied.

The reason for this relation is as follows: we consider the linear one dimensional problem

$$p'' + f'(w(t))p = q(t), \quad t \in (0, \infty), \quad p(0) = p'(0) = 0 \quad \text{(0.18)}$$

The solution to this equation is given by

$$p(t) \;=\; w'(t) \int_0^t \frac{d\tau}{w'(\tau)^2} \int_0^\tau w'(s)\, q(s)\, ds \; . \qquad \text{(0.19)}$$

If q is a bounded function, p will be bounded if and only if

$$\int_0^\infty q(t)\, w'(t)\, dt = 0.$$

We consider the next order term in the separated form

$$u = w(t) + \epsilon\phi, \phi \sim \Psi(y)p(t)$$

Then $\Psi(y) = H$ and p satisfies

$$p^{''} + f^{'}(w)p = w^{'}(t)$$

which does not have a solution satisfying

$$p(0) = 0, p^{'}(0) = 0$$

This forces $H \equiv 0$.

But if we only ask for a solution satisfying

$$p(0) = 0$$

then we only need

$$H \equiv Constant$$

Let us take the function h to have the following form:

$$h(y) = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3$$

Then we find, by successive approximation

$$h_0 = c_0$$

$$\Delta_\Gamma h_1 + |A_\Gamma|^2 h_1 = \sum_{i=1}^{8} k_i^3$$

$$\Delta_\Gamma h_2 + |A_\Gamma|^2 h_2 = \sum_{i=1}^{8} k_i^4$$

$$\Delta_\Gamma h_2 + |A_\Gamma|^2 h_3 = |A_\Gamma|^4$$

where $\mathcal{J}[h] = \Delta_\Gamma h + |A_\Gamma|^2$ is the Jacobi operator.

At ∞, $\Delta_\Gamma \sim \Delta, |A_\Gamma|^2 \sim r^{-2}$. Thus we have a Hardy Type operator

$$\mathcal{J} \sim \Delta + \frac{a(\theta)}{r^2}$$

We will show that $\mathcal{J}$ has indicial roots r^{-2} and r^{-3}. On the other hand,

$$k_i = O(\frac{1}{r})$$

$$\sum_i k_i^3 = O(\frac{1}{r^3})$$

$$\sum_i k_i^4 = O(\frac{1}{r^4})$$

$$|A_\Gamma|^4 = O(\frac{1}{r^4})$$

Let us formally choose solutions to be

$$u = u_0(t) + \epsilon^2 \phi$$

Then one gets

$$S[u] = \Delta u + f(u) = S[u_0] + \epsilon^2 S^{'}[u_0](\phi) + h.o.t$$

where in a local chart

$$S^{'}[u_0] \sim \Delta \phi + f^{'}(u_0)\phi$$

and

$$S[u_0] \sim \epsilon^2 (\Delta_\Gamma \alpha + |A_\Gamma|^2 \alpha)$$

However,

$$\partial_\nu u_0 \sim w_0^{'}(0) + \epsilon^2 \beta(y)$$

Hence the condition that $\partial_\nu u = C$ becomes

$$\frac{\partial \phi}{\partial y} = \beta(y) + C \tag{0.20}$$

Since $\beta(y) \to 0$ and we expect that ϕ decays, this forces $C = 0$.

Thus ϕ satisfies

$$
\begin{aligned}
\Delta\phi + f'(w(t))\phi &= \Delta_\Gamma h + |A_\Gamma|^2 h(y)\, w'(t) + h.o.t. \quad \text{in } \mathbb{R}^9_+, \\
\phi(y,0) &= 0 \qquad \text{for all} \quad y \in \mathbb{R}^8, \\
\partial_t \phi(y,0) &= \beta(y) \quad \text{for all} \quad y \in \mathbb{R}^8.
\end{aligned} \tag{0.21}
$$

Thus the linear problem becomes Given the boundary value $\beta(y)$, we need to find a $\alpha(y) = \Delta h + |A|^2 h$ such that ϕ satisfies the constraint.

A Cauchy Problem

Let us write $\mathbb{R}^9_+ := \mathbb{R}^8 \times (0,\infty)$. We consider the problem of finding, for given functions $g(y,t)$, $\beta(y)$, a solution (α,ϕ) to the problem

$$\begin{aligned} \Delta\phi + f'(w(t))\phi &= \alpha(y)\, w'(t) + g(y,t) \quad \text{in } \mathbb{R}^9_+, \\ \phi(y,0) &= 0 \qquad \text{for all} \quad y \in \mathbb{R}^8, \\ \partial_t\phi(y,0) &= \beta(y) \quad \text{for all} \quad y \in \mathbb{R}^8. \end{aligned} \tag{0.22}$$

This is a Cauchy problem for a linear elliptic operator. Can we solve Cauchy problem for elliptic equations?

Solvability of Cauchy's Problem

Lemma

Given β and g such that

$$\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^8)} + \|g\|_{C_0^{0,\sigma}(\mathbb{R}^9_+)} < +\infty$$

there exists a solution

$$(\phi, \alpha) \in C_0^{2,\sigma}(\mathbb{R}^9_+) \times C_0^{0,\sigma}(\mathbb{R}^8)$$

of Problem $(\mathbf{??})$ *that defines a linear operator of the pair* (β, g)*, satisfying the estimate*

$$\|\phi\|_{C_0^{2,\sigma}(\mathbb{R}^9_+)} + \|\alpha\|_{C_0^{0,\sigma}(\mathbb{R}^8)} \leq C\,[\,\|\beta\|_{C_0^{1,\sigma}(\mathbb{R}^8)} + \|g\|_{C_0^{0,\sigma}(\mathbb{R}^9_+)}\,]. \tag{0.23}$$

Solvability for the Jacobi operator

We consider the linear problem

$$\mathcal{J}_\Gamma[h] := \Delta_\Gamma h + |A_\Gamma(y)|^2 h = \mathrm{g}(y) \quad \text{in } \Gamma. \tag{0.24}$$

The following result was established in del Pino-Kowalczyk-Wei 2011

Lemma

Let $4 < \nu < 5$. There exists a positive constant $C > 0$ such that if g *satisfies*

$$\|\mathtt{r}^\nu\, g\|_{L^\infty(\Gamma)} \, < \, +\infty$$

*then there is a unique solution of equation (**??**) such that* $\|\mathtt{r}^{\nu-2}\, h\|_{L^\infty(\Gamma)} < +\infty$*. This solution satisfies*

$$\|\mathtt{r}^{\nu-2}\, h\|_{L^\infty(\Gamma)} \, \leq \, C\, \|\mathtt{r}^\nu\, g\|_{L^\infty(\Gamma)}\, .$$

The proof of this result is based on the construction of explicit barriers, using the fact that the surfaces Γ and Γ_0 are uniformly close for r large. Barriers constitute an appropriate tool to solve Problem (**??**) since $\mathcal{J}_\Gamma$ satisfies maximum principle, as it follows from the presence of a positive bounded function in its kernel. In fact, we have that

$$\mathcal{J}_\Gamma[\frac{1}{\sqrt{1+|\nabla F|^2}}] = 0.$$

In the current setting we need to consider right hand sides with decay of order at most $O(r^{-4})$, the prototypes being $\mathrm{g} = \sum_{i=1}^{3} k_i^3$ and $\mathrm{g} = \sum_{i=1}^{3} k_i^4$. It is not possible in general to obtain a suitable barrier in the setting of the above proposition when $\nu \leq 4$. We have however the validity of Lemma **??** below which will suffice for our purposes.

The closeness of the surfaces allows us to replace Γ by Γ_0.

We can compute explicitly the operator $\mathcal{J}_{\Gamma_0}$ as follows. Let us consider the first variation of mean curvature measured along vertical perturbations of the graph Γ_0, namely the linear operator $H'(F_0)$ defined by

$$H'(F_0)[\phi] \;:=\; \frac{d}{dt}H(F_0+t\phi)\,|_{t=0}$$

$$=\;\nabla\cdot\left(\frac{\nabla\phi}{\sqrt{1+|\nabla F_0|^2}}-\frac{(\nabla F_0\cdot\nabla\phi)}{(1+|\nabla F_0|^2)^{\frac{3}{2}}}\nabla F_0\right).$$

Then we have the relation

$$\mathcal{J}_{\Gamma_0}[h]=H'(F_0)[\phi],\quad \text{where}\quad \phi(x')=\sqrt{1+|\nabla F_0(x')|^2}\,h(x',F(x')). \tag{0.25}$$

For vertical perturbations $\phi = \phi(r, \theta)$ of Γ_0, it is straightforward to compute

$$H'(F_0)[\phi] := \tilde{L} := \tilde{L}_0 + r^{-4}\tilde{L}_1, \tag{0.26}$$

with

$$\begin{aligned} L_0(\phi) \;=\; & \frac{1}{r^7 \sin^3(2\theta)} \big\{ (9g^2\, \tilde{w}_0 r^3 \phi_\theta)_\theta \;+\; (r^5 {g'}^2\, \tilde{w}_0 \phi_r)_r \\ & \qquad - 3(gg'\, \tilde{w}_0 r^4 \phi_r)_\theta - 3(gg'\, \tilde{w}_0 r^4 \phi_\theta)_r \big\}. \end{aligned} \tag{0.27}$$

Crucial in the proof of Lemma **??**, as in the arguments that follow below is the presence of explicit solutions that separate variables for the operator L_0. Let us consider the equation

$$L_0(r^\beta q(\theta)) = \frac{p(\theta)}{r^{4-\beta}}, \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \tag{0.28}$$

By a direct computation we get the following explicit formula for a solution $q(\theta)$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

$$q(\theta) = g^{\frac{\beta}{3}}(\theta)\,\Big[-\frac{1}{9}\int_{\frac{\pi}{4}}^{\theta} g^{-\frac{2}{3}}(\,9g^2+g'^2\,)^{\frac{3}{2}}\,\frac{ds}{\sin^3(2s)}\int_s^{\frac{\pi}{2}} p(\tau)g^{-\frac{\beta+4}{3}}(\tau)\sin^3(2$$

This integral in not convergent if $\beta \geq 0$ and $p(\frac{\pi}{4}) \neq 0$.

Lemma

(a) *Let $p(\theta)$ be a smooth function, even with respect to $\pi/4$, namely*

$$p(\frac{\pi}{2} - \theta) = p(\theta) \quad \textit{for all} \quad \theta \in (0, \frac{\pi}{4}).$$

Then there exists a smooth function $h(r, \theta)$ with the same symmetry, that satisfies, for some $\mu > 0$,

$$\mathcal{J}_{\Gamma_0}[h] = \frac{p(\theta)}{r^4} + O(r^{-4-\mu}) \quad \textit{as } r \to +\infty, \tag{0.30}$$

and

$$\|\mathtt{r}^2 (\log \mathtt{r})\, h\,\|_{L^\infty(\Gamma_0)} \;<\; +\infty.$$

Lemma

(b) *Let* $p(\theta)$ *be a smooth function, odd with respect to* $\pi/4$*, namely*

$$p(\frac{\pi}{2}-\theta) = -p(\theta) \quad \textit{for all} \quad \theta \in (0, \frac{\pi}{4}).$$

Then there exists a smooth function $h(r,\theta)$ *with the same symmetry, such that for some* $\mu > 0$*,*

$$\mathcal{J}_{\Gamma_0}[h] = \frac{p(\theta)}{r^3} + O(r^{-4-\mu}) \quad \textit{as } r \to +\infty, \tag{0.31}$$

and

$$\|\mathtt{r}\, h\,\|_{L^\infty(\Gamma_0)} \; < \; +\infty,$$

and, in addition,

$$|\nabla_{\Gamma_0} h|^2 \; = \; O(r^{-4-\mu}) \quad \textit{as } r \to +\infty. \tag{0.32}$$

The above lemmas can be used to solve

$$\Delta_\Gamma h_1 + |A_\Gamma|^2 h_1 = \sum_{i=1}^{8} k_i^3$$

since

$$\sum_{i=1}^{8} k_i^3 = \frac{g(\theta)}{r^3}$$

because of the symmetry

$$\Delta_\Gamma h_2 + |A_\Gamma|^2 h_2 = \sum_{i=1}^{8} k_i^4$$

since

$$\sum_{i=1}^{8} k_i^4 = \frac{p}{r^4}$$

$$\Delta_\Gamma h_2 + |A_\Gamma|^2 h_3 = |A_\Gamma|^4$$

since

$$|A_\Gamma|^4 = \frac{p}{r^4}$$

Thank You