NEW TRENDS IN NONLINEAR ELLIPTIC EQUATIONS

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1 Overview

The theory of elliptic equations expanded during the last decades in very different directions with various applications. Progress was made in particular in regularity theory, uniqueness techniques, fully nonlinear issues, qualitative properties of solutions, singularities, etc.. One can mention in particular several domains like Navier-Stokes equations, homogenization, calculus of variations, Ginzburg-Landau problems, singular perturbations and the Allen-Cahn equations. The workshop brought together actors of these different fields who shared their experience of the different techniques involved. Very challenging questions were concerned. A total of 25 lectures, 50-minutes long, were given by leading experts from all over the world (USA, Canada, France, Switzerland, Italy, Germany, Spain, Chile, Japan, Israel) as well as young promising researchers who are either at the postdoctoral level or in the early stages of a tenure-track position.

2 Scientific background

The original idea beyond the workshop was to bring together mathematicians working in two different themes: Ginzburg-Landau theory and stationary Navier-Stokes equations. By the Ginzburg-Landau theory we mean both the stationary equation, known also as Allen-Cahn equation,

$$-\Delta u = (1 - u^2)u \,, \tag{1}$$

and the vectorial version, that appears in Superconductivity, which in the simplest case (in the absence of magnetic field) takes the form

$$-\Delta u = (1 - |u|^2)u\,, \tag{2}$$

with u taking its values in $\mathbb{R}^2$.

Regarding the scalar problem, important progress was made recently towards the resolution of the famous De Giorgi conjecture [8]. It states that if u is a solution of (1) in $\mathbb{R}^n$ which is monotone increasing in one direction, say $\frac{\partial u}{\partial x_n} > 0$, then the level sets of u must be hyperplanes, at least for $n \leq 8$. The conjecture was completely resolved in dimension two (by Ghoussoub and Gui [15]) and three (by Ambrosio and Cabre [2]). The weak form of the De Giorgi conjecture, in which the assumption that $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$ is made, was resolved by Savin for $n \leq 8$ (see [24]). On the other hand, Del Pino, Kowalczyk and Wei [9] have recently shown that for $n \geq 9$ solutions whose level sets are not hyperplanes do exist. The De Giorgi conjecture in its strong form is still open for dimension $4 \leq n \leq 8$. Let us notice that J. Wei was one of the participants in the workshop and although his talk did not treat directly the De Giorgi conjecture, his talk on Serrin's overdetermined problem presented some analogy between the problems. In fact, Serrin's overdetermined problem deals with solutions on domains of epi-graph type,

$$\Omega = \{x \in \mathbb{R}^n \,:\, x_n > \varphi(x_1, \ldots, x_{n-1})\},$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function, namely,

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \{x_n = \varphi(x_1, \ldots, x_{n-1})\}, \\ \dfrac{\partial u}{\partial \nu} = \text{const on } \{x_n = \varphi(x_1, \ldots, x_{n-1})\}. \end{cases} \tag{3}$$

As an example of such an analogy, Theorem 1.1 in [25] states that in dimension $n = 2$, for certain functions f, existence of a solution to (3) implies that Ω is a half-plane, and up to an isometry, $u(x) = g(x \cdot e)$ for some unit vector e.

The vectorial Ginzburg-Landau equation (2) on $\mathbb{R}^2$ arises by a "blow-up" procedure of solutions $\{u_\varepsilon\}$ to the equation

$$-\Delta u_\varepsilon = \frac{(1 - |u_\varepsilon|^2)}{\varepsilon^2} u_\varepsilon \tag{4}$$

on a bounded domain Ω in $\mathbb{R}^2$. The special case of minimizers of the Ginzburg-Landau type energy

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \tag{5}$$

is particularly interesting. The study of the asymptotic behavior of the minimizers (and even critical points) of the energy E_ε when ε goes to zero, under Dirichlet boundary condition $g : \partial\Omega \to S^1$ was carried-up in the book [3] and many subsequent works. In the more physical Ginzburg-Landau model in Superconductivity one has to replace the energy (5) by the energy

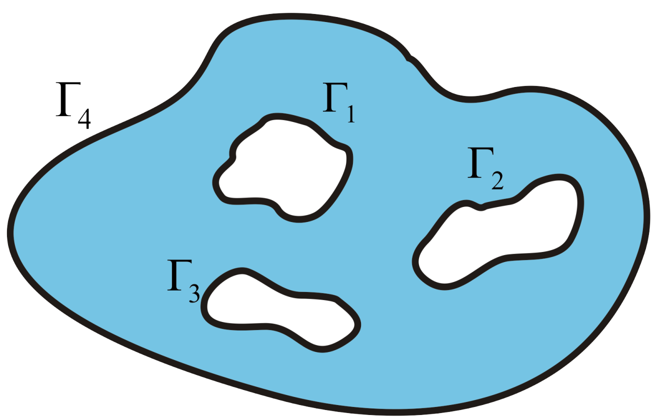
$$G_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |h - h_{ex}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\,, \tag{6}$$

where h_{ex} is the applied magnetic field. The resulting Euler-Lagrange equations satisfied by the minimizers (or more generally critical points) (u, A) of (6) take the form:

$$\begin{cases} -(\nabla_A u)^2 u = \dfrac{1}{\varepsilon^2}(1-|u|^2)u \text{ in } \Omega, \\ -\nabla^\perp h = (iu, \nabla_A u) \text{ in } \Omega, \\ h = h_{ex} \text{ on } \partial\Omega, \\ \nu\cdot\nabla_A u = 0 \text{ on } \partial\Omega. \end{cases} \tag{7}$$

One of the main difficulties in the analysis of (4) and (7) is the control of the “vortices”, the set on which $|u|$ is close to zero. The “vortex-balls construction” that was developed independently by Sandier and Jerrard (both participants in the workshop) played a major role in the important progress made in recent years in the mathematical analysis of solutions to (7), notably by the works of Sandier-Serfaty (see [23]).

The second main theme, stationary Navier-Stokes, was supposed to be around the Leray problem. This challenging problem was introduced in 1933 by J. Leray in a seminal paper. Suppose that a fluid is occupying a bounded domain as the one in the figure below with an outside boundary and insides multiple boundaries.



The stationary Navier–Stokes system in $\Omega \subset \mathbb{R}^n, n = 2, 3$, in to find a vector valued function $\mathbf{u}$ satisfying

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div }\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \end{cases}$$

- $\mathbf{u}(x)$ – is the velocity field, $\mathbf{u} \cdot \nabla = u_i \partial_{x_i} = \sum_i u_i \partial_{x_i}$,
- $p(x)$ – denotes the pressure of the fluid,
- ν – is the constant coefficient of viscosity,
- $\mathbf{a}(x)$ – the boundary value,
- $\mathbf{f}(x)$ – the external force.

The incompressibility of the fluid (div $\mathbf{u} = 0$) implies a necessary condition for the solvability of the problem above. One has indeed

$$0 = \int_\Omega \text{div } \mathbf{u} dx = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, dS = \sum_{j=1}^{N} \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, dS := \sum_{j=1}^{N} F_j,$$

where $\mathbf{n}$ is a unit vector of the outward normal to $\partial\Omega$. For a long time the solvability of the problem above was proved only under the condition

$$F_j = \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, dS = 0, \quad j = 1, 2, ..., N.$$

Recently Korobkov, Pileckas and Russo were able to solve this problem in some particular cases, but the problem is still open in its full generality (see [18], [17]). Even due to cancellations of speakers related to the Navier-Stokes theme, there were interesting discutions about these issues. Caraballo and Chipot gave two lectures regarding this theme. Quite a few lectures were also given on Ginzburg-Landau equations and related topics (by Almog, Jerrard, Mironescu, Sandier, Shafrir, Davila), but also a large variety of other topics in the elliptic theory were represented, including: Liouville-type equations, the p-laplacian, constant non-local mean curvature, extremum eigenvalues problems, Helmholtz equations with sign changing coefficients, prescribed Jacobian inequalities and many more. As a result, the topic of the workshop was less focused than expected, but instead very diversified, presenting a wide spectrum of the current state of the research in Elliptic Partial Differential Equations. A more specific description is given in the next sections.

3 Navier Stokes Theory

T. Caraballo [6] introduced several methods to analyze the long time behaviour of solutions to 2D Navier-Stokes models. Allowing very general delay terms, the problem becomes nonautonomous in general. For this reason he considered the theory of nonautonomous pullback attractors.

M. Chipot [7] presented existence results for the stationary non homogeneous Navier-Stokes problem in a two dimensional symmetric domain having a semi-infinite outlet. Under the symmetry assumptions on the domain, boundary value and external force he proved the existence of at least one weak symmetric solution, without any restriction on the size of the fluxes.

4 Ginzburg-Landau theory

R. Jerrard Starting from the Ginzburg-Landau model, he derived an effective free energy functional for nearly-parallel vortex filaments. As a consequence, he obtained the existence of solutions of the Ginzburg-Landau equations, in certain scaling regimes, possessing a collection of vortex filaments minimizing this effective energy.

P. Mironescu considered maps $u : \Omega \to \mathbb{S}^1$ having some Sobolev regularity $u \in W^{s,p}$:

a) either such maps need not have a phase φ with the same regularity as u
b) or the phase φ exists but is not controlled by the norm of u.

In case a), factorization allows to write each such u as $u = v\, w$, where v lifts and w is "smoother" than u. Case b) occurs only in dimension one.

The common theme of the proofs of the above results is the geometric detection of the energy concentration of manifold-valued maps.

E. Sandier brought together approaches of Dal Maso-Modica and Alberti-Müller to provide a framework for the analysis of multiscale problems. He applied it to a random version of a problem studied by Alberti-Müller. The approach generalizes the one in previous work of S. Serfaty and himself on the Ginzburg-Landau functional.

I. Shafrir [4, 19] Certain Sobolev spaces of S^1-valued functions can be written as a union of disjoint classes. It is interesting to study the distances between these classes. In his talk, based on a joint work with Brezis and Mironescu, he concentrated on classes in $W^{1,1}(\Omega, S^1)$, where Ω is a simply connected domain in $\mathbb{R}^N$, $N \geq 2$. He presented estimates for the minimal distance as well as the Hausdorff distance between different classes.

J. Davila was interested in singularity formation for the harmonic map flow from a two dimensional domain into the sphere. He showed that for suitable initial conditions the flow develops a type 2 singularity at some point in finite time, and that this is stable under small perturbations of the initial condition. All these results hold without any symmetry assumptions.

Y. Almog [1] investigated the time-dependent Ginzburg-Landau equations in the presence of strong currents, however weaker than the critical current where the normal state losses its stability. In the large κ limit, he proved that the superconductivity order parameter is exponentially small in a significant part of the domain, and small in the rest of it.

5 Scalar nonlinear elliptic equations

J. Wei [10, 25] In 1971, Serrin proved that the only bounded domain for which the overdetermined problem

$$\Delta u + f(u) = 0, u > 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

$$\partial_\nu u = C \text{ on } \partial\Omega$$

admits a solution is the ball. In 1997, Berestycki, Caffarelli and Nirenberg considered the unbounded domain case, and proposed the following conjecture: If Serrin's problem admits a solution and Ω^c is connected, then Ω is either a half space, a cylinder $B \times \mathbb{R}^{N-k}$, or complement of a ball or cylinder. In the talk was discussed positive and negative answers to this conjecture. In particular, when Ω is an epigraph $\Omega = \{x_N > \varphi(x^{'})\}$, it was shown that

(1) BCN conjecture is always true when $N = 2$,
(2) BCN conjecture is true when $3 \leq N \leq 8$ if $\frac{\partial u}{\partial x_N} > 0$,
(3) BCN conjecture is false when $N \geq 9$.

A key observation is the connection between this problem and a one-phase free boundary problem.

B. Kawohl gave a survey of solved and unsolved problems on the first nonconstant Neumann-eigenfunction for the p-Laplacian. Among other things, he studied the limit of the problem when p goes to ∞.

M. Fazly [12] provided classification and symmetry results for certain local and non-local elliptic PDEs with power type nonlinearities. It was the occasion to review the background on standard methods and ideas developed over the past couple of decades. Then he turned to present monotonicity formulas, Liouville theorems and one-dimensional symmetry properties.

E. Yanagida gave a survey of results of removable and non-removable singularities in elliptic and parabolic equations. Among the problems mentioned were the Laplace equation, the Lane-Emden equation, the absorption equation $u_t = \Delta u - u^p$, Fujita equation $u_t = \Delta u + u^p$.

6 Nonlinear parabolic equations and systems

Y. Du [11] was interested in the nonlinear parabolic problem

$$u_t - \Delta u = f(u)(x \in \mathbb{R}^N, t > 0),\ u(x,0) = u_0(x)\ \ (x \in \mathbb{R}^N),$$

where $u_0 \in L^\infty(\mathbb{R}^N)$ is nonnegative and has compact support, f is a smooth function satisfying $f(0) = 0$. He wanted to study how much of the long-time dynamics of this problem is determined by the corresponding elliptic problem

$$-\Delta u = f(u),\ u \geq 0\ \ (x \in \mathbb{R}^N).$$

M. Musso constructed global unbounded solutions for the critical nonlinear heat equation on a bounded smooth domain satisfying zero Dirichlet boundary conditions. Given an integer k, and given any set of k distinct points of the domain, which satisfy a certain condition involving Green's function of the domain, she found a positive solution for the critical heat equation blowing up at exactly those k points as time goes to infinity.

D. Kinderlehrer [16] spoke about the Poisson-Nernst-Planck system of equations used to model ionic transport is interpreted as a gradient flow for the Wasserstein distance and a free energy in an appropriate space of probability measures. The interaction term between the species arising from the Gauss law is singular which gives rise to some challenging issues. He gave a description of this situation attempting to maintain a minimal technical level including the basic format of the Wasserstein-type implicit scheme.

7 Liouville-type problems

G. Wolansky considered an extension of the Keller-Segel model to several cells populations. He reviewed some of the results considered before in the literature and, in particular, considered the case of conflict between two populations, that is, when population one attracts population two, while, at the same time, population two repels population one. This assumption leads to a new functional inequality which generalizes the Moser-Trudinger inequality. As an application of this inequality he derived sufficient conditions for the existence of steady states corresponding to solutions of an elliptic Liouville system.

A. Poliakovsky [22], motivated by the study of non-abelian Chern Simons vortices of non-topological type in Gauge Field Theory, analysed the solvability of some Liouville-type system in presence of singular sources. He was able to identify necessary and sufficient conditions which ensure the radial solvability of this system.

8 Calculus of variations

B. Dacorogna proved (Darboux theorem) that if ω_m is the standard symplectic form and f is a symplectic form, then one can find a diffeomorphism φ, with optimal regularity, satisfying

$$\varphi^* (\omega_m) = f \quad \text{and} \quad \delta [\varphi \lrcorner \omega_m] = 0$$

provided that f is a small perturbation of ω_m. He then apply the above result to the so-called symplectic decomposition. The connections with mass transportation and the Monge-Ampère equation were emphasized.

O. Kneuss [13] spoke about existence of solution to the prescribed Jacobian inequality coupled with a Dirichlet condition, namely

$$\begin{cases} \det \nabla\phi \geq f & \text{a.e. in } \Omega \\ \phi = \mathrm{id} & \text{on } \partial\Omega \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth connected open set, $f : \Omega \to \mathbb{R}$ and where $\phi : \Omega \to \mathbb{R}^2$ is the unknown. He showed that this prescribed Jacobian inequality in the plane admits – unlike the prescribed Jacobian equation – a bi-Lipschitz solution in case of right-hand sides of class L^∞.

9 Other topics

X. Cabré [5] was interested in hypersurfaces of $\mathbb{R}^N$ with constant nonlocal (or fractional) mean curvature. First he proved the nonlocal analogue of the Alexandrov result characterizing spheres as the only closed embedded hypersurfaces in $\mathbb{R}^N$ with constant mean curvature. Then by the moving planes method he establishes the existence of periodic bands or "cylinders" in $\mathbb{R}^2$ with constant nonlocal mean curvature and bifurcating from a straight band.

H-M. Nguyen [21] devoted his talk to various properties and applications of the Helmholtz equations with sign changing coefficients. These equations are used to model negative index materials which are artificial structures whose refractive index are negative over some frequency range. The study of these equations faces two difficulties. First the ellipticity and the compactness are lost in general due to the changing sign coefficients. Second, the localized resonance, i.e., the fields blow up in some regions and remain bounded in some others as the loss (the viscosity) goes to 0, might occur.

J. Fischer [14] developed a large-scale regularity theory of higher order for divergence-form elliptic equations with heterogeneous coefficient fields a in the context of stochastic homogenization. Under the assumptions of stationarity and slightly quantified ergodicity of the ensemble, he derived a $C^{k,\alpha}$-"excess decay" estimate on large scales and a $C^{k,\alpha}$-Liouville principle for any $k \geq 2$: For a given a-harmonic function u on a ball B_R, he showed that its energy distance on some ball B_r to the space of a-harmonic functions that grow at most like a polynomial of degree k has the natural decay in the radius r, at least above some minimal (random) radius r_0. His results rely on the existence of higher-order correctors for the homogenization problem, which are established by an iterative construction.

C. Wang [20] described a uniqueness result of absolute minimizers of Hamiltonian functions $H(x,p)$, provided

(i) H is lower semicontinuous, and $H(x,p)$ is convex in p;

(ii) $0 = H(x,0) \leq H(x,p)$ and $\cup_x\{p : H(x,p) = 0\}$ is contained in a hyperplane of R^n;

(iii) $H(x,p)$ is uniformly coercive in p.

S. Mardare considered a Bingham flow in a domain which is periodic in one direction. Interesting is the asymptotic behaviour of the solution to the stationary Bingham problem as the length 2ℓ of the domain (in the periodic direction) goes to infinity. The main result states that the velocity of the fluid converges strongly in the H^1-norm to the solution of a Bingham problem in the infinite periodic domain. Nevertheless, the speed of the convergence is much lower than the one obtained for the (linear) Stokes problem.

10 Outcome of the workshop

The workshop was a great opportunity for researchers to be updated in the last developments in nonlinear elliptic (and also parabolic) problems, to meet and interact with other researchers, either close or at least realed to their own research. It gave many opportunities to continue existing collaborations and to start new ones. The large number of open problems that were presented in the talks will certainly encourage research in the area by participants and their Ph.D. students or Postdoctoral researchers. One of the successes of the workshop was in giving the opportunity to several young promising researchers (Fazly, Fischer, Kneuss, Poliakovsky) to present their research in an important scientific meeting.

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