Computability in Geometry and Analysis

Mark Braverman (Princeton), Michael Yampolsky (Toronto)

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The workshop brought together experts and junior mathematicians in various areas of Analysis, Dynamics, Geometry, and Topology where computability and complexity ideas play a key role. Theory of computation of analytic objects is an emerging field of mathematics which has seen much exciting progress recently. The talks by the workshop participats outlined the main current directions of study, which we describe below.

1 Computability problems in topology of knots

Two beautiful talks on the subject were given by M. Lackenby (Oxford) and N. Dunfield (UIUC). We briefly survey the field below.

1.1 Computational complexity of knot genus

A fundamental property of a null-homologous knot K in a closed 3-manifold Y is the minimal genus $g(K)$ of an embedded surface in Y with boundary K. In the 1960s, Haken used normal surface theory to give an algorithm which computes $g(K)$, opening the door to a whole subfield and the discovery of algorithms for determining a wide range of topological properties. However, algorithms based on normal surface theory are usually exponential-time both in theory and practice [16]. Moreover, in some cases the underlying problems have been shown to be in complexity classes which are thought to be fundamentally difficult. For example, consider the following decision problem:

KNOT GENUS. *Given an integer g_0 and a null-homologous knot K embedded in the 1-skeleton of a triangulation $\mathcal{T}$ of a closed 3-manifold, is $g(K) \leq g_0$?*

Agol, Hass, and W. Thurston showed that KNOT GENUS is NP–complete [1], and so has the same computational complexity as e.g. the Traveling Salesman Problem. The conjecture that P $\neq$ NP thus implies that there is no algorithm for KNOT GENUS which runs in time polynomial in the size of $\mathcal{T}$.

However, when $b_1(Y) = 0$, for instance $Y = S^3$, then KNOT GENUS should be considerably easier than NP–complete, perhaps even solvable in polynomial time. Here are three pieces of evidence for this. First, as discussed below in Section 1.2, Hirani and Dunfield showed that a closely related but more geometric problem can be solved in polynomial time when $b_1(Y) = 0$ despite being NP–complete in general. Second, Lackenby, using an approach of Agol, has recently proved

Theorem. [(Lackenby 2015)] *When $b_1(Y) = 0$, then KNOT GENUS is in coNP.*

The Agol-Lackenby approach uses Gabai's sutured manifold hierarchies to bound the genus from below.

If P is the class of decision problems for which there are polynomial-time algorithms, then the standard conjectures are that the known containments P $\subset \big($NP $\cap$ coNP$\big) \subset$ NP are all proper. Thus, as KNOT GENUS is in NP $\cap$ coNP when $b_1(Y) = 0$, there is the potential for an algorithm in this special case which is substantially faster than for an NP-complete problem. Third, using a very different approach relying on results in gauge theory and arithmetic geometry, Kuperberg had previously proved that for $Y = S^3$ the $g_0 = 0$ case of KNOT GENUS is in coNP [34].

1.2 Minimal spanning area

Returning to the general setup, let K be a null-homologous knot in a closed 3-manifold Y with no restriction on $b_1(Y)$. Let M be a simplicial complex triangulating the exterior $Y \setminus \operatorname{int}\big(N(K)\big)$ of K, and assign to each 2-simplex in M a positive real number which we refer to as its area. Consider the set $\mathcal{F}$ of simplicial maps $f \mapsto (S, \partial S) \to (M, \partial M)$ where S is an orientable surface with boundary and $f_*\big([\partial S]\big)$ generates the kernel of $H_1(\partial M) \to H_1(M)$. Thus, when we collapse ∂M back to K, the image $f(S)$ is a (possibly non-embedded) surface homologically bounding K. The areas of 2-simplices of M naturally define the area of each $f \in \mathcal{F}$. Consider:

LEAST SPANNING AREA. *Given $A_0 \in \mathbb{N}$ and the exterior M of a null-homologous knot $K \subset Y$ is there an $f \in \mathcal{F}$ with $\operatorname{Area}(f) \leq A_0$?*

The work of [1] shows this problem is NP-complete. Despite this, Hirani and Dunfield proved the following, which made Dunfield conjecture in his talk KNOT GENUS is in P for knots in S^3.

Theorem. [19] *For manifolds with $b_1(Y) = 0$, the LEAST SPANNING AREA problem can be solved in polynomial time.*

1.3 Random knots

Dunfield also described his experimental work with a graduate student Malik Obeiden on random prime knots with 100 to 1,000 crossings, both to probe algorithmic complexity in practice and to better understand the properties of random knots in the spirt of [21, 20]. Their initial findings give an evidence of linear growth (with little spread) with respect to crossing number of the following invariants: hyperbolic volume (slope ≈ 2), knot genus (slope ≈ 0.25), and bridge number (slope ≈ 0.15). One pattern that demands explanation: these knots have triangulations where most of the tetrahedra are "fat" in the hyperbolic structure, that is, have volumes near that of the regular ideal one.

2 Algorithmic randomness and computable Ergodic theory

D. Hirschfeldt (Chicago) gave a beatiful introduction to Algorithmic Randomness. Building on his talk, C. Rojas (Andres Bello) and J. Avigad (Carnegie Mellon) described applications of Computability to Ergodic Theory, which is an area of research relating measurable dynamics to theoretical computer science. Its general goal is to understand in a precise mathematical way the theoretical simulation and computation of the long term behavior of dynamical systems. Among the objects describing this limiting behavior, of particular interest are invariant measures and generic points which provide a complete statistical description of the system. Their computability and complexity properties – in the sense of theoretical computer science – and its relationship to the dynamical, geometrical and analytical properties of the system, are the main questions of the subject. The tools and techniques vary according to the nature of the different systems involved, but they have all one thing in common: one has to deal with rigorous notions of computability for infinite objects. This is done in the field known as "Computable Analysis" ([36, 5, 47]). The main idea here being that an object x is *computable* if there exists an algorithm (a Turing Machine) which, upon input $\varepsilon \in \mathbb{Q}$, will produce a "finite" object describing x at accuracy ε with respect to some suitable notion of distance. In what follows we recall the most relevant known results on the subject, and state some directions of active research.

2.1 Invariant measures

Computability of invariant measures in the rigorous sense of computable analysis is a very recent topic of research. One way to prove computability of an invariant measure is to use some known strong statistical properties of the system, like for example decay of correlations as done in [24]. The algorithms obtained in this way, however, usually depend on some finite information that might be not known, and therefore they are non-uniform. The most general technique to obtain uniform algorithms consists, very roughly speaking, on finding a list of suitable semi-decidable conditions that together characterize a particular invariant measure of interest. The computability of this measure then follows from general computability considerations involving the effectiveness of certain compact sets. These results have the advantage of being simple and quite general with all the needed assumptions made explicit. They have a wide range of application including several different classes of systems like hyperbolic systems (see [26]) or rational functions (see [8]). On the other hand, they are not well suited for a complexity (in time or space) analysis, so it is not clear if they can be implemented and used in practice. The rigorous framework in which they are proved, however, allows to see them as a study about the theoretical limits of (Turing-)computation of invariant measures and, in fact, also negative results can be obtained. For example, there exists computable systems for which every invariant measure is non-computable (see [26]), and systems having measures of maximal entropy, all of which are non computable ([8]). There are also examples of computable invariant measures which are a computable combination of finitely many ergodic measures all of which are non computable (see [31]). An important direction of research here is the study of the robustness properties of these kind of non computable phenomena. In [14] it is shown for example that the addition of a small amount of noise at each step of the evolution of the system is sufficient to destroy the non computability of the ergodic measures – the noise turns them from non computable into computable. However, a more fine analysis of how persistent can the non computable phenomenon be, is still lacking. Another important open research direction is the analysis of the computational resources required to achieve the computation of an invariant measure and the way this complexity relates to the dynamics. In particular, how sensitive is the computational complexity with respect to small changes to the dynamics. Some progress was made in [14] where the authors show that for noisy systems, if the noise itself is not a source of additional complexity, then the invariant measures can be computed efficiently, namely in time polynomial in the number of bits required to specify the precision of the computation.

2.2 Effective ergodic theorems, randomness and pseudorandomness

The main question here is to understand the effectiveness of the rate of convergence in the ergodic theorems. It has long been known that this rate can be arbitrarily slow [33, 32]. However, the point here is not directly about the speed of convergence, but rather about the information required to algorithmically extract a bound on this rate [3, 4]. These effective results are important because they allow to obtain more concrete information about how the finite structures underlying the ergodic behavior are constructed. For example, they are useful in developing algorithms to compute points exhibiting good statistical properties ([24, 27]). They have played a role in recent results in number theory and combinatorics ([2, 44]). They are also important in the theory of algorithmic randomness ([18]) as a tool to calibrate the degree of randomness required by points in order to satisfy the ergodic theorem (or other almost everywhere convergence results) with respect to specific observables. For example, the understanding achieved on the computable content in the ergodic theorem for ergodic measures versus non ergodic ones, gave rise to a series of results ([45, 46, 23, 25, 28, 40, 7, 22]) culminating in a sharp characterization of generic points in terms of algorithmic randomness.

3 Computability questions in Complex Analysis

3.1 Computability of the Riemann mapping

Theoretical aspects of computability of the Riemann mapping were discussed by I. Binder (Toronto). We briefly summarize the discussion below.

An open set U in the plane is called *lower-computable* if there exists a computable sequence of rational balls whose union exhausts the set U. Similarly, a closed set K is lower-computable if rational balls which intersect K can be computably enumerated.

Let Ω be a simply-connected proper subdomain of $\mathbb{C}$ and let $w_0 \in \Omega$. The celebrated Riemann Mapping Theorem states that there exists a unique conformal homeomorphism

$$g : \Omega \to \mathbb{D} \text{ such that } g(w_0) = 0 \text{ and } g'(w_0) > 0. \qquad (1)$$

We will denote $f \equiv g^{-1} : \mathbb{D} \to \Omega$. Hertling showed in [30] that g and f are computable if and only if: Ω is a lower-computable open set, $\partial\Omega$ is a lower-computable closed set, and $w_0 \in \Omega$ is a computable point.

The computational complexity of the Riemann mapping were addressed by Binder, Braverman, and Yampolsky in [9]. They showed, in particular:

Theorem. [9] *Suppose there is an algorithm A that given a simply-connected domain Ω with a linear-time computable boundary, a point $w_0 \in \Omega$ with$\mathrm{dist}(w_0, \partial\Omega) > \frac{1}{2}$ and a number n, computes $20n$ digits of the conformal radius $f'(0)$), then we can use one call to A to solve any instance of a #SAT(n) with a linear time overhead.*

In other words, $\#P$ is poly-time reducible to computing the conformal radius of a set.

Note, that any algorithm computing values of the uniformization map f will also compute the conformal radius with the same precision, by Koebe Distortion Theorem. Conversely,

Theorem. [9] *There is an algorithm A that computes the uniformizing map in the following sense:*

Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. Assume that the boundary of a simply connected domain Ω, $\partial\Omega$, $w_0 \in \Omega$, and $w \in \Omega$ are provided to A by an oracle. Then A computes $g(w)$ with precision n with complexity PSPACE(n).

Rettinger later observed that the proof of [9] actually gives a better complexity bound, $\#P$.

Binder also described the work [10], in which he, Rojas, and Yampolsky developed the computable version of the Carathéodory Theory of prime ends, proving a computable version of the Carathéodory Theorem.

A special case of this theorem states that f continuously extends to the unit circle if and only if $\partial\Omega$ is locally connected. The computable version of this proved in [**?**] relies on the definition of the *Carathéodory modulus*. Namely, a non-decreasing function $\eta(\delta)$ is called the Carathéodory modulus of Ω if $\eta(\delta) \to 0$ as $\delta \to 0$ and if for every crosscut γ with $\mathrm{diam}(\gamma) < \delta$ we have $\mathrm{diam}\, N_\gamma < \eta(\delta)$. Here N_γ is the component of $\Omega \setminus \gamma$ not containing w_0.

It was shown in [10], that the Carathéodory extension of $f : \mathbb{D} \to \Omega$ is computable iff f is computable and there exists a computable Carathéodory modulus of Ω. Furthermore, it was shown that there exists a domain Ω with computable Carathéodory modulus but no computable modulus of local connectivity.

3.2 Computational aspects of the conformal mapping: Zipper algorithm

S. Rohde's (Washington) was based in part on his joint work with Don Marshall. The best known algorithm for computing the conformal mapping is Zipper algorithm proposed by Marshall [35]. Its convergence was proven by Rohde and Marshall in [42]. Rohde described the algorithm, together with the related Loewner equation and conformal welding. He explained how the zipper algorithm can be viewed as a discretization of the Loewner differential equation, and how this discretization can be implemented to produce an approximation to a given conformal map as a composition of a large number of conformal maps onto half-planes slit by a hyperbolic geodesic (or straight line). He sketched the proof of the convergence of the algorithm, a key idea being the use of Jorgensen's theorem about the hyperbolic convexity of Euclidean discs in planar domains.

The celebrated Schramm-Loewner Evolution SLE is obtained by using Brownian motion as the driving function for the Loewner equation. Rohde gave a brief overview of some highlights regarding SLE, some of their path properties, and a comparison to the corresponding deterministic results. He then proceeded to the seemingly unrelated topic of Grothendieck dessins d'enfants and the associated Belyi functions: Every connected graph drawn on the sphere can be realized (up to homeomorphism) as the preimage $f^{-1}(L)$ under a rational map with at most three critical values (Belyi function), where L is a line segment joining two of the critical points. In the special case when the graph is a tree (no cycles), one of the critical points can be normalized to be infinity, and the Belyi function is a polynomial (Shabat polynomial). The computation of Belyi functions has spurred a lot of research and led to many publications in computational algebra, but the algebraic methods fail as soon as the degree of the polynomial exceeds about 10. Rohde explained how the zipper algorithm can be modified to numerically approximate Shabat polynomials and their trees when the degree is very large (even in the thousands!). The method also allows to re-construct a dendrite from its lamination, and Rohde illustrated this in the setting of quadratic Julia sets. He then proceeded to a discussion of random trees, an object of very high current interest: He explained how a bijection between trees and their Dyck paths leads to the Aldous Continuum Random Tree in the scaling limit, and how this CRT arises

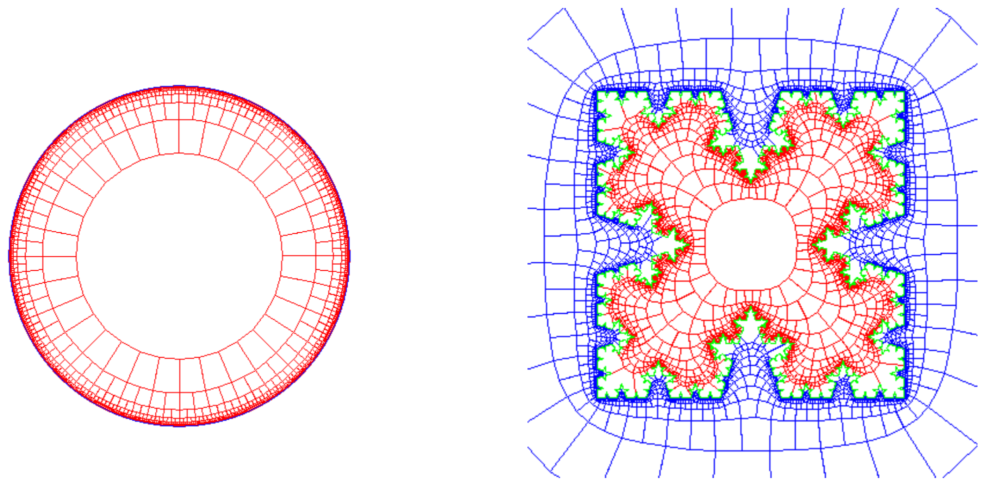

Figure 1: Zipper algorithm at work: Carleson grid in the unit disk (left) and its image under the Riemann map inside a "snowflake" domain.

as a building block of the scaling limit of large random maps, the so-called Brownian map. He concluded by discussing partial results to the existence question of the distributional limit of the Shabat trees when the degree tends to infinity.

4 Decidability of equivalence problems in topological dynamics

4.1 Decidability of the Thurston equivalence problem

N. Selinger (Stony Brook) spoke about the Thurston equivalence problem for branched covering maps. A *Thurston mapping* $f : S^2 \to S^2$ is a branched covering of a finite topological degree $d > 1$, and such that the orbits of the branched points are finite. f is *Thurston equivalent* to g if there exist homeomorphisms ψ_1, $\psi_2 : S^2 \to S^2$ such that $(\psi_2)^{-1} \circ f \circ \psi_1 = g$, and there is an isotopy between ψ_1 and ψ_2 which does not move the orbits of the branched points of the coverings. A celebrated theorem of Thurston [17] answers the question when a Thurston mapping is equivalent to a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The criterion of Thurston is formulated in terms of non-existence of certain finite collections of homotopy classes of loops on the sphere, known as *Thurston obstructions*.

A question, which had remained open for some time, was whether there exists an algorithm which given a finite description of the branched covering f can answer whether a rational map R exists or not. We resolved this long-standing problem in a recent work of M. Braverman, S. Bonnot, and M. Yampolsky [11]. A more general problem is:

Thurston Equivalence Problem. *Is there an algorithm which decides whether two Thurston mappings f and g are equivalent or not?*

N. Selinger and M. Yampolsky [43] we have made significant progress towards resolving this questions by proving that any Thurston mapping can be constructively canonically geometrized. This allowed them to partially resolve the problem of comparing two maps, yet complete solution requires further work.

4.2 Conjugacy problem for expanding maps

V. Nekrashevych (Texas A& M) reported on his current work on the conjugacy problem for expanding maps. Let (X, d) be a compact metric space. A map $f : X \to X$ is said to be *expanding* if there exist $\epsilon > 0$ and $L > 1$ such that $d(f(x), f(y)) \geq Ld(x, y)$ for all $x, y \in X$ such that $d(x, y) < \epsilon$.

Expanding covering maps $f : X \to X$ can be given (modulo topological conjugacy) by a finite amount of information. For example, one can use the fact that they are *finitely presented*: such a map is conjugate to a quotient of a shift of finite type by an equivalence relation which is also a shift of finite type.

If X is locally connected and connected, then there is also a natural group associated with $f : X \to X$ which is a complete invariant. For $t \in X$ consider the tree of preimages T_t with the set of vertices equal to the (formal) disjoint union of the sets$f^{-n}(t)$ for $n \geq 0$. Here a vertex $v \in f^{-n}(t)$ is connected to $f(v) \in f^{-(n-1)}(t)$. If γ is a path from t_1 to t_2, then for every $v \in f^{-n}(t_1)$ there exists a unique lift γ_v of γ by f^n starting at v. Denote the end of γ_v by $S_\gamma(v)$. Then $S_\gamma : T_{t_1} \to T_{t_2}$ is an isomorphism. It also induces a homeomorphism of the boundaries $S_\gamma : \partial T_{t_1} \to \partial T_{t_2}$.

Fix $t \in X$, and consider two finite sets $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_n\}$ of vertices of T_t such that ∂T_t is equal to the disjoint union $\bigsqcup_{i=1}^n \partial T_{v_i}$ and to the disjoint union $\bigsqcup_{i=1}^n \partial T_{u_i}$. Choose paths γ_i from v_i to u_i. Then union of the maps S_{γ_i} is a homeomorphism of ∂T_t with itself. Denote by $\mathcal{V}_f$ the set of all such homeomorphisms.

Nekrashevych reported the following result:

Theorem. *The set $\mathcal{V}_f$ is a group. It is finitely presented, its derived subgroup is simple. It is a complete invariant of topological conjugacy: two dynamical systems are topologically conjugate if and only if the corresponding groups are isomorphic as abstract groups.*

He then posed a natural question, whether the problem of topological conjugacy of expanding self-coverings is algorithmically solvable. While the question is open, there is hope that it can be resolved using group-theoretic methods.

5 Words in linear groups, random walks, automata, and P-recursiveness

I. Pak (UCLA) described his current work with S. Garrabrant. An integer sequence $\{a_n\}$ is called *polynomially recursive*, or *P-recursive*, if it satisfies a nontrivial linear recurrence relation of the form

$$(*) \qquad q_0(n)a_n + q_1(n)a_{n-1} + \ldots + q_k(n)a_{n-k} = 0,$$

for some $q_i(x) \in \mathbb{Z}[x]$, $0 \leq i \leq k$. The study of P-recursive sequences plays a major role in modern Enumerative and Asymptotic Combinatorics. They have *D-finite* (also called *holonomic*) generating series

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} a_n t^n,$$

and various asymptotic properties.

Let G be a group and $\mathbb{Z}[G]$ denote its group ring. For every $g \in G$ and $u \in \mathbb{Z}[G]$, denote by $[g]u$ the value of u on g. Let $a_n = [1]u^n$, which denotes the value of u^n at the identity element. When $G = \mathbb{Z}^k$ or $G = F_k$, the sequence $\{a_n\}$ is known to be P-recursive for all $u \in \mathbb{Z}[G]$. Maxim Kontsevich asked in 2014 whether $\{a_n\}$ is always P-recursive when $G \subseteq \mathrm{G}L(k, \mathbb{Z})$. Pak and Garrabrant gave a negative answer to this question:

Theorem. *There exists an element $u \in \mathbb{Z}[SL(4, \mathbb{Z})]$, such that the sequence $\{[1]u^n\}$ is not P-recursive.*

Pak described two proofs of the theorem. The first proof is completely self-contained and based on ideas from computability. Roughly, one gives an explicit construction of a finite state automaton with two stacks and a non-P-recursive sequence of accepting path lengths. One then converts this automaton into a generating set $S \subset \mathrm{S}L(4, \mathbb{Z})$. The key part of the proof is a new combinatorial lemma giving an obstruction to P-recursiveness.

The second proof is analytic in nature, and is the opposite of being self-contained. The problem is interpreted in a probabilistic language, a number of advanced and technical results in Analysis, Number Theory, Probability, and Group Theory are used to derive the theorem.

6 Other complexity questions in groups with applications

M. Sapir (Vanderbilt) gave a beautiful talk on various computing devices (Turing machines, S-machines, Minsky machines) used in dealing with algorithmic problems in group theory.

A. Nabutovsky (Toronto) described his joint work with B. Lishak. In his earlier works [39, 37, 38], Nabutovsky described the geometric complexity of the space of Riemannian metrics on a manifold of dimension $d \geq 5$ which arises from undecidability of word problem in the fundamental group, and related geometric complexity phenomena resulting from non-computability in groups. The work with Lishak extends these results to dimension $d = 4$. Namely, Nabutovsky and Lishak proved that: 1) There exist infinitely many nontrivial codimension one "thick" knots in $\mathbb{R}^5$; 2) For each closed four-dimensional smooth manifold M and for each sufficiently small positive ϵ the set of isometry classes of Riemannian metrics with volume equal to 1 and injectivity radius greather than ϵ is discon- nected; 4) For each closed four-dimensional PL-manifold M there exist arbitrarily large values of N such that some two triangulations of M with $< N$ simplices cannot be connected by any sequence of $< M(N)$ bistellar transformations, where $M = 2^{2^{2^{\cdots^{2}}}}$ ($[\log_2 N] -$ const times).

7 Information and communication complexity

A.Garg (Princeton) gave a talk on communication and information complexity. Two-party communication complexity is the study of how much communication two parties need to exchange to compute a function of their private inputs. It is one of the few models of computation in which strong unconditional lower bounds are known and has applications throughout complexity theory, for example to lower bounds on circuits, streaming algorithms, data structures etc. In the past 5-10 years, the study of information complexity has greatly advanced the state-of-the-art knowledge about communication complexity. Information complexity is the study of how much information two parties need to exchange to compute a function of their private inputs. It has helped in tackling hard questions in communication complexity, for example direct sum and direct product theorems. The best known direct sum and direct product theorems for randomized communication complexity are proven via information complexity. Roughly, the current best direct sum theorem [6] says that the amount of communication required to compute n independent copies of a function is at least $\Omega(\sqrt{n})$ times the amount of communication required for one copy. Furthermore, it has also been proven [15] that the success probability decays exponentially in n. Such statements are known as direct product theorems. Improving the known direct sum and direct products theorems is equivalent to the compression question: whether uninformative conversations can be compressed down to their information content. It is known that the compression question does not have a positive answer in its full generality [29], but it still largely remains an open problem. Information complexity also has other applications within communication complexity and complexity theory in general. For example, it can be used to pin down the exact communication complexity of disjointness up to low order terms [13]. Also the techniques used for proving direct product theorems in communication complexity have helped in getting new proofs (and improvements) of the parallel repetition theorem [41, 12].

8 A complexity theory of constructible functions and sheaves

S. Basu (Purdue) gave a talk on constructible functions and sheaves. Constructible functions and more generally constructible sheaves play a very important role in algebraic geometry with many applications, including in theory of D-mocules, algebraic theory of partial differential equations, and even in more applied areas such as computational geometry and signal processing. In his talk Basu described an approach towards developing a complexity theory for these objects, which generalizes the Blum-Shub-Smale model over $\mathbb{R}$. More precisely, he introduced a class of sequences simple constructible sheaves, that could be seen as the sheaf-theoretic analog of the Blum-Shub-Smale class $\mathrm{P}_{\mathbb{R}}$. He also defined a hierarchy of complexity classes of sheaves mirroring the polynomial hierarchy, $\mathrm{PH}_{\mathbb{R}}$ in the B-S-S theory. He proved a singly exponential upper bound on the topological complexity of the sheaves in this hierarchy mirroring a similar result in the BSS setting. He obtained as a result an algorithm with singly exponential complexity for a sheaf-theoretic variant of the real quantifier elimination problem. Finally, he posed the natural sheaf-theoretic analogs of the classical P vs NP question, and also discussed a connection with Toda's theorem from discrete complexity theory in the context of constructible sheaves.

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