Uniqueness of minimizers of general weighted least gradient problems

Amir Moradifam

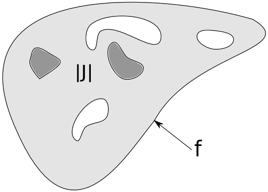
(University of California, Riverside)

Banff International Research Station

June 2015

Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set.



Question: Can the shape and location of the perfectly conducting and insulating inclusions and the isotropic conductivity σ outside of the inclusions be determined from the knowledge of $(f, |J|)$?

Notations

- O_0: Insulating inclusions
- O_∞: Perfectly conducting inclusions
- $O_\infty = \cup_{i=1}^{\infty} O_\infty^i$: the partition of O_∞ into its connected open components

The forward problem

Assume the voltage f is imposed on the boundary of Ω then the corresponding voltage potential u is the unique $H^1(\Omega)$ solution of the following equation.

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0, & \text{in}\, \Omega \setminus \overline{O_0 \cup O_\infty}, \\ \nabla u = 0, & \text{in}\;\; O_\infty, \\ u|_+ = u|_-, & \text{on}\;\; \partial(O_0 \cup O_\infty), \\ \int_{\partial O_\infty^i} \sigma \frac{\partial u}{\partial \nu}|_+ ds = 0, & i = 1, 2, ..., \\ \frac{\partial u}{\partial \nu}|_+ = 0, & \text{on}\; \partial O_0, \\ u|_{\partial\Omega} = f. \end{cases} \tag{1}$$

Hence

$$(O_0, O_\infty, \sigma) \;\; \Rightarrow \;\; u \;\; \Rightarrow \;\; |J|$$

The Inverse Problem

$$(f, |J|) \;\; \Rightarrow \;\; u \;\; \Rightarrow \;\; (O_0, O_\infty, \sigma)$$

Assume there are no inclusions. Then the forward problem simplifies to

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \;\text{ in }\; \Omega \\ u \mid_{\partial\Omega} = f. \end{cases}$$

Since

$$\sigma = \frac{|J|}{|\nabla u|},$$

$$\begin{cases} \nabla \cdot (|J| \frac{\nabla u}{|\nabla u|}) = 0 \;\text{ in }\; \Omega \\ u = f \;\text{ on }\; \partial\Omega. \end{cases} \tag{2}$$

This equation does not have a unique solution in general.

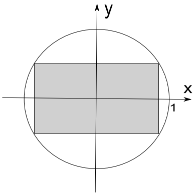
Sternberg-Ziemer example

Let D be the unit disk in $\mathbb{R}^2$. Then for every $\lambda \in [-1,1]$

$$u_\lambda = \begin{cases} 2x^2-1, & \text{if} \quad |x| \geq \sqrt{\frac{1+\lambda}{2}}, \quad |y| \leq \sqrt{\frac{1-\lambda}{2}}, \\ \lambda, & \text{if} \quad |x| \leq \sqrt{\frac{1+\lambda}{2}}, \quad |y| \leq \sqrt{\frac{1-\lambda}{2}}, \\ 1-2y^2, & \text{if} \quad |x| \leq \sqrt{\frac{1+\lambda}{2}}, \quad |y| \geq \sqrt{\frac{1-\lambda}{2}}. \end{cases} \tag{3}$$

is a viscosity solution of the 1−Laplacian equation

$$\begin{cases} \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) = 0, & \text{in} \ \ D, \\ u = x^2 - y^2, & \text{on} \ \ \partial D. \end{cases}$$



Energy functional

Sternberg and Ziemer (1994) proved that u_0 is the solution of

$$u_0 = \text{argmin}\{\int_D |\nabla v|, \;\; v \in BV(D), \;\; \text{and} \;\; v|_{\partial D} = x^2 - y^2\}. \qquad (4)$$

We expect the voltage potential u to be the unique minimizer of the functional

$$F(v) = \int_\Omega |J||\nabla v| dx, \qquad (5)$$

on

$$\Theta = \{v \in BV(\Omega) : \;\; v \mid_{\partial\Omega} = f\}.$$

Unique determination

Theorem (M., A. Nachman, A. Tamasan (2012))

Assume that $(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times L^2(\Omega)$ *is generated by some unknown inclusions* O_0 *and* O_∞ *and unknown piecewise* C^α *conductivity* σ*. Then the corresponding voltage potential u is a solution of the problem*

$$u = argmin\{\int_\Omega |J||\nabla v| : v \in BV(\Omega), \;\; v|_{\partial\Omega} = f\}, \tag{6}$$

and if u^* *is another minimizer, then* $u^* = u$ *in*

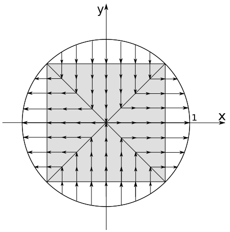
$$\Omega \setminus \{x \in \Omega : \;\; |J| = 0\}.$$

Sternberg-Ziemer example

Let $I_c = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \times (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and define

$$\sigma = \begin{cases} \frac{1}{4|x|}, & \text{if} \quad |x| \geq \frac{1}{\sqrt{2}}, \quad |y| \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{4|y|}, & \text{if} \quad |x| \leq \frac{1}{\sqrt{2}}, \quad |y| \geq \frac{1}{\sqrt{2}}, \end{cases} \quad \text{on} \;\; D \setminus \bar{I}_c. \tag{7}$$

If the voltage $f = x^2 - y^2$ is imposed on ∂D, then the voltage potential $u = u_0$ and the following current density J with $|J| \equiv 1$ will be induced.



Equipotential sets are minimal surfaces

Theorem (M., A. Nachman, A. Tamasan (2011))

Almost every equipotential set of u has zero mean curvature (i.e. are minimal surfaces) in the metric $g = |J|^{2/(n-1)} I$*. Indeed, the equipotential surfaces are critical points for the functional*

$$\mathcal{A}(\Sigma) = \int_{\Sigma} |J| dS. \tag{8}$$

Sketch of the uniqueness proof

1. Almost every level set of u stays away from the perfectly conducting inclusions.
2. By a regularity result of De Giorgi we conclude that these level sets are C^1-hypersurfaces.
3. Let u^* is another minimizer, then

$$\frac{\nabla u(x)}{|\nabla u(x)|} = \frac{\nabla u^*(x)}{|\nabla u^*(x)|} \text{ a. e. on } (\Omega \setminus \bar{I}) \cap \{x \in \Omega : |\nabla u^*| \neq 0\}. \tag{9}$$

4. (9) $\Rightarrow u^* = c$ on almost every level set of u.
5. By Alexander's duality theorem in algebraic topology we conclude almost every level set of u intersects $\partial\Omega$.
6. Since $u = u^* = f$ on $\partial\Omega$, $u = u^*$ on $a.e.$ level set of u. Hence $u = u^*$.

Anisotropic conductivities

Anisotropic conductivities

Let

$$\sigma \in C^\alpha(\Omega, Mat(\mathbb{R}, n))$$

be a symmetric matrix-valued anisotropic conductivity and u is the voltage potential

$$\nabla \cdot (\sigma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

Question: Can σ be determined from the knowledge of (J, f)?
Assume

$$\sigma = c(x)\sigma_0, \tag{10}$$

where σ_0 is known. Indeed σ_0 can be determined by Diffusion Tensor Imaging (DTI). The conductivity σ can be determined from the knowledge of (J, f, σ_0).

Unique determination

For $b \in (L^2(\Omega))^n$

$$|b|_{\sigma_0} := \sqrt{\sigma_0 b \cdot b}$$

Theorem (N. Hoell, M. , A. Nachman (2013))

Assume that (f, J) is generated by some unknown inclusions O_0 and O_∞ and unknown piecewise C^α anisotropic conductivity σ. Then the corresponding voltage potential u is a solution of the problem

$$u \in argmin\{\int_\Omega |J|_{\sigma_0^{-1}} |\nabla v|_{\sigma_0} : v \in W^{1,1}(\Omega), \;\; v|_{\partial\Omega} = f\}, \tag{11}$$

and if u^* is another minimizer, then $u^* = u$ in

$$\Omega \backslash \{x \in \Omega : \;\; |J| = 0\}.$$

Uniqueness in $BV(\Omega)$

Define

$$\varphi(x,\xi) = a(x)\left(\sum_{i,j=1}^{n} \sigma_0^{ij}(x)\xi_i\xi_j\right)^{1/2}. \tag{12}$$

For $u \in BV(\Omega)$ we define the **weighted total variation** of u (with respect to φ) in Ω as

$$\int_\Omega |Du|_\varphi = \sup_{b\in\mathfrak{B}_a} \int_\Omega u\nabla\cdot b \; dx, \tag{13}$$

where

$$\mathfrak{B}_a = \{b \in (L^\infty(\Omega))^n : \text{ spt}(b) \text{ is compact in } \Omega, \; \nabla\cdot b \in L^n(\Omega), \\ \text{and } |b|_{\sigma_0^{-1}} \le |J|_{\sigma_0^{-1}} \text{ a.e. in } \Omega\}.$$

Unique determination in $BV(\Omega)$

Theorem (N. Hoell, M. , A. Nachman (2013))

Assume that (f, J) is generated by some unknown inclusions O_0 and O_∞ and unknown piecewise C^α anisotropic conductivity σ. Then the corresponding voltage potential u is a solution of the problem

$$u \in argmin\{\int_\Omega |Dv|_\varphi : v \in BV(\Omega), \quad v|_{\partial\Omega} = f\}, \tag{14}$$

and if u^* is another minimizer, then $u^* = u$ in

$$\Omega \backslash \{x \in \Omega : \quad |J| = 0\}.$$

Equipotential sets are minimal surfaces

Theorem (N. Hoell, M. , A. Nachman (2013))

Almost every equipotential set of u has zero mean curvature (i.e. are minimal surfaces) in the metric

$$g_{ij} = (|\sigma_0||J|^2_{\sigma_0^{-1}})^{\frac{1}{n-1}}(\sigma_0^{-1})_{ij}$$

Indeed, the equipotential surfaces are critical points for the functional

$$\mathcal{A}(\Sigma) = \int_\Sigma \sqrt{\sigma_0^{-1} J \cdot J} dS.$$

General existence and uniqueness results for

$$\text{argmin}\{\int_\Omega a|Du| : v \in BV(\Omega), \;\; v|_{\partial\Omega} = f\}.$$

Strong barrier condition

Assume $E \subset \mathbb{R}^n$ has finite perimeter, then weighted perimeter of E in Ω is defined by

$$P_a(E;\Omega) := \int_\Omega a|D\chi_E|. \tag{15}$$

Definition (Strong barrier condition)

We say that Ω satisfies the strong barrier condition if for every $x_0 \in \partial\Omega$ and $\epsilon > 0$ sufficiently small if $V \subset \Omega$ minimizes $P_a(\,\cdot\,;\mathbb{R}^n)$ in

$$\{W \subset \mathbb{R}^n : W \setminus B(\epsilon, x_0) = \Omega \setminus B(\epsilon, x_0)\}, \tag{16}$$

then

$$\partial V \cap \partial\Omega \cap B(\epsilon, x_0) = \emptyset.$$

This means that moving $\partial\Omega$ inward by small amounts decreases the perimeter of Ω.

Existence and uniqueness

Theorem (R.L. Jerrard, M. , A. Nachman (2015))

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying the strong barrier condition. Suppose $a \in C^{1,1}(\Omega)$ is bounded away from zero and $f \in C(\Omega^c)$. Then the problem

$$argmin\{\int_\Omega a|Du| : v \in BV(\Omega), \ \ v|_{\partial\Omega} = f\}. \tag{17}$$

has a unique minimizer in $BV(\Omega)$.

Proposition (R.L. Jerrard, M. , A. Nachman (2015))

For any $\alpha < 1$, there exists a bounded smooth domain $\Omega \subset \mathbb{R}^n$ satisfying the strong barrier condition, $f \in C(\Omega^c)$, and a function $a \in C^{1,\alpha}(\Omega)$ such that the problem (17) has infinitely many minimizers.

A weak maximum principle for minimal surfaces

Theorem (R.L. Jerrard, M. , A. Nachman (2015))

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying the strong barrier condition. Suppose $a \in C^{1,1}(\Omega)$ is bounded away from zero. If $E_1, E_2 \subset \mathbb{R}^n$ are area minimizing in Ω and $E_1 \setminus \Omega \subset\subset E_2 \setminus \Omega$, then

$$E_1 \subset E_2.$$

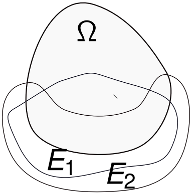


Figure: Failure of weak maximum principle

Proof of the weak maximum principle

1. $F_1 = E_1 \cap E_2$ and $F_2 = E_1 \cup E_2$ are both area minimizing and F_1 lies in one side of F_2.
2. It follows from a deep regularity result of Schoen, Simon, and Almgren (1977) that if E is area minimizing in Ω then

$$\begin{cases} H^{n-3}(\text{sing}(\partial E) \cap \Omega) < \infty & \text{if } \ n \geq 3 \\ \text{sing}(\partial E) \cap \Omega = \emptyset, & \text{if } \ n = 2. \end{cases} \tag{18}$$

3. First we show that Hausdorff dimension of $\partial E_1 \cap \partial E_2$ is at least $n - 2$.
4. Since $x \in \partial E_1 \cap \partial E_2 \subset \partial F_1 \cap \partial F_2$, by (18)

$$dim_H(reg(\partial F_1) \cap reg(\partial F_2)) \geq (n-2).$$

5. For every $x \in reg(\partial F_1) \cap reg(\partial F_2)$, it follows from an strong maximum principle for fully non-linear elliptic equations that F_1 and F_2 coincide in neighbourhood of x. This leads to a contradiction.

A convergent algorithm to solve

$$u = \operatorname{argmin}\{\int_\Omega |J||\nabla v| : v \in H^1(\Omega), \;\; v|_{\partial\Omega} = f\}.$$

Fenchel-Rockafellar duality

Let H_1 and H_2 be real Hilbert spaces and consider the problem

$$(P) \qquad \min_{u \in H_1} \{E(Lu) + G(u)\},$$

where $L : H_1 \to H_2$ is a bounded linear operator. Then the problem of (P) can be written as

$$(D) \qquad -\min_{b \in H_2} \{G^*(-L^*b) + E^*(b)\}.$$

Moreover, if b is any solution of the dual problem, then the entire set of solutions of the primal problem is obtained as

$$\partial G^*(-L^*b) \cap L^{-1}\partial E^*(b). \tag{19}$$

The dual problem

Let $u_f \in H^1(\Omega)$ with $u_f \mid_\Omega = f$. Then our weighted minimization problem can be written as

$$(P) \qquad \inf_{v \in H_0^1(\Omega)} \int_\Omega |J||\nabla v + \nabla u_f|.$$

The dual problem is

$$(D) \quad \sup\{< \nabla u_f, b >: \ b \in (L^2(\Omega))^n, \ |b(x)| \leq |J(x)| \ a.e. \text{ and } \nabla \cdot b \equiv 0\}.$$

Current density J is the unique solution of the dual problem

Proposition (M., A. Nachman, A. Timonov (2011))

$$\inf_{v \in H_0^1(\Omega)} \int_\Omega |J||\nabla v + \nabla u_f|$$

$$=$$

$$\sup\{< \nabla u_f, b >: \quad b \in (L^2(\Omega))^n, \quad |b(x)| \leq |J(x)| \quad a.e. \quad and \quad \nabla \cdot b \equiv 0\}$$

and the current density J induced by the voltage potential f on $\partial\Omega$ is the unique solution of the dual problem.

Primal and Dual problems

Let

$$E(d) = \int_\Omega |J||d + \nabla u_f| \text{ and } G \equiv 0.$$

Then primal problem can be written as

$$(P) \qquad \min_{u \in H_0^1(\Omega)} \{E(\nabla u) + G(u)\}.$$

Also

$$(D) \qquad -\min_{b \in (L^2(\Omega))^n} \{E^*(b) + G^*(-\nabla \cdot b)\}.$$

Solution of the Dual problems

Since J is the solution of the dual problem

$$0 \in \partial E^*(J) + \partial[G^* o(\nabla \cdot)](J). \tag{20}$$

Let $A := \partial E^*(J)$ and $B := [G^* o(\nabla \cdot)]$. The (20) can be written as

$$0 \in A(J) + B(J),$$

where A and B are maximal monotone operators.

Douglas-Rachford algorithm for maximal monotone operators

To solve

$$0 \in A(J) + B(J) \tag{21}$$

we apply Douglas-Rachford algorithm. This algorithm produces two sequences p_k and x_k such that

$$p_k \rightharpoonup J \text{ and } x_k \rightharpoonup \hat{x} \in A(J) = \partial E^*(J) = \{\nabla u\}.$$

Thus

$$p_k \rightharpoonup J \text{ and } x_k \rightharpoonup \nabla u.$$

Douglas-Rachford Algorithm

Theorem (Lions and Mercier (1979), Svaiter (2010))

Let H be a Hilbert space and A, B be maximal monotone operators and assume that a solution of (23) exists. Then, for any initial elements x_0 and p_0 the sequences p_k and x_k generated by the following algorithm

$$\begin{aligned} x_{k+1} &= J_A(2p_k - x_k) + x_k - p_k \\ p_{k+1} &= J_B(x_{k+1}), \end{aligned} \tag{22}$$

converges weakly to some $\hat{x}$ and $\hat{p}$ respectively. Furthermore, $\hat{p} = J_B(\hat{x})$ and $\hat{p}$ satisfies

$$0 \in A(\hat{p}) + B(\hat{p}). \tag{23}$$

$$J_P = (Id + P)^{-1}$$

Evaluation of the resolvents

Lemma

Assume x_k and p_k are given and let u^{k+1} and d^{k+1} and be the minimizers of the functionals

$$I_1(u) = \| (2p_k - x_k) + \nabla u \|^2,$$

$$I_2(d) = E(d) + \frac{1}{2} \| x_{k+1} - d \|^2,$$

respectively. Then

$$J_A(2p_k - x_k) = \nabla u^{k+1} + 2p_k - x_k, \quad and$$

$$p_{k+1} = J_B(x_k) = x_{k+1} - d^{k+1}.$$

The algorithm [M., A. Nachman, A. Tamasan (2011)]

Let $u_f \in H^1(\Omega)$ with $u_f|_{\partial\Omega} = f$, and initialize $b^0, d^0 \in (L^2(\Omega))^n$. For $k \geq 1$:

1. Solve
$$\Delta u^{k+1} = \nabla \cdot (d^k(x) - b^k(x)), \quad u^{k+1}|_{\partial\Omega} = f.$$
2. Compute
$$d^{k+1} := \begin{cases} \max\{|\nabla u^{k+1} + b^k| - |J|, 0\} \frac{\nabla u^{k+1} + b^k}{|\nabla u^{k+1} + b^k|} & \text{if } |\nabla u^{k+1}(x) + b^k(x)| \neq 0, \\ 0 & \text{if } |\nabla u^{k+1}(x) + b^k(x)| = 0. \end{cases}$$
3. Let
$$b^{k+1}(x) = b^k(x) + \nabla u^{k+1}(x) - d^{k+1}(x).$$

Convergence of the algorithm

Theorem (M., A. Nachman, A. Timonov (2011))

The sequences b^k, d^k, and u^k produced by the above algorithm converge weakly to J, ∇u, and u, respectively.

Numerical simulations

Numerical simulation

To simulate the internal data $|J|$ we use a CT (Computed Tomography) image of human abdomen rescaled to a realistic range of tissue conductivities.

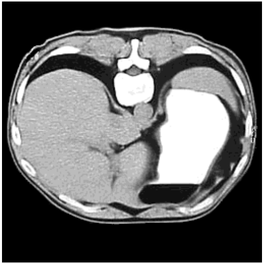


Figure: Origional image (left) and reconstructed image with 60 iterations (right).

Conductivity reconstruction

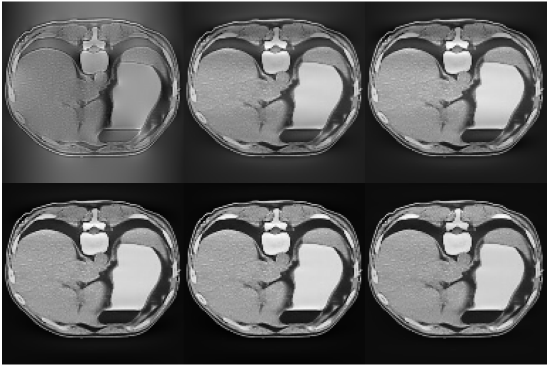


Figure: Conductivity reconstruction with the boundary condition $f(x, y) = y$ for $N = 1, 5, 10, 30, 50, 100$ iterations.

Perfectly conducting and insulating inclusions

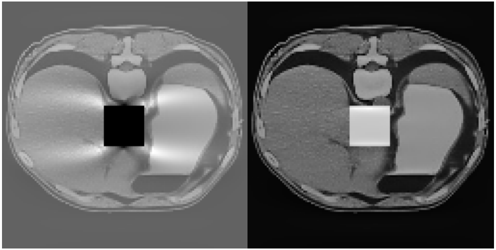
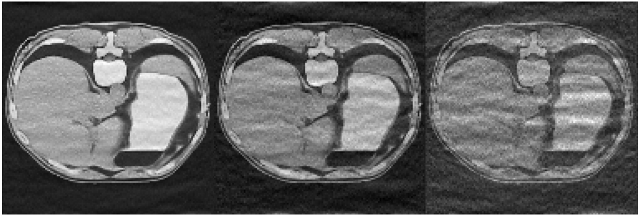


Figure: Reconstruction in the presence of the perfectly conducting (right) and insulating (left) inclusions.

Effect of noise

Numerical errors for 100 iterations.

Low Noise (Level=0.01)	Moderate Noise (Level=0.035)	Higher Noise (Level=0.06)
0.026	0.080	0.152



References

- Existence and uniqueness of minimizers of general least gradient problems, (with R.L. Jerrard and A. Nachman), J. Reine Angew. Math., to appear.
- Current Density Impedance Imaging with an Anisotropic Conductivity in a Known Conformal Class, SIAM J. Math. Anal., 46 (2014), 3969-3990 (with N. Hoell and A. Nachman).
- A convergent algorithm for the hybrid problem of reconstructing conductivity from minimal interior data, Inverse Problems 28 (2012), 084003 (with A. Nachman and A. Timonov).
- Conductivity imaging from one interior measurement in the presence of perfectly conducting and insulating inclusions, SIAM J. Math. Anal., 44 (2012), 3969-3990 (with A. Nachman and A. Tamasan).