Current Density Impedance Imaging with Complete Electrode Model

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Outline

In motivation
Hybrid methods in Inverse Problems
Current density based EIT
Acquiring the interior data

The forward problem
The Complete Electrode Model

The Inverse Problem
Characterization of non-uniqueness
Phase retrieval
Restoring uniqueness
A numerical algorithm and experiment
Conclusions
Coupled Physics Imaging Methods

Combine high contrast & high resolution

- **Elastography**: elastic waves & ultrasound/MRI $\Rightarrow$ stiffness
- **Thermo/PhotoAcoustic**: UV light & sound $\Rightarrow$ embedded acoustic sources
- **AcoustoOptics**: light & sound $\Rightarrow$ absorption and scattering

**Coupled Physics Electrical Impedance Tomography**

- Current density impedance imaging **CDII**: Joy & Nachman since 2002, Seo et al. 2002
- **MREIT** ($B_z$-methods): Seo et al. since 2003
- Impedance acoustic: Scherzer et al. 2009
- Lorentz force driven EIT: Ammari et al. since 2013
- ...
Interpretation of the voltage potential along \( \Gamma \)

Interior measurement of the magnitude of the current density

Reconstruction
Current density tracing inside an object

Figure: Courtesy: Joy’s group, U Toronto
Magnetic resonance data: \( M : \Omega \to \mathbb{C} \)

\[
M_{\pm}(x, y, z_0) = M(x, y, z_0) \exp(\pm i \gamma B_z(x, y, z_0) T + i \varphi_0)
\]
Aquiring the interior data

One MR scan ⇒ longitudinal component $B_z$ (along gantry) of the magnetic field $\mathbf{B} = (B_x, B_y, B_z)$

$$B_z(x, y, z_0) = \frac{1}{2\gamma T} \Im \log \left( \frac{M_+(x, y, z_0)}{M_-(x, y, z_0)} \right)$$

- MREIT (Seo at al. since 2003): Does $B_z$ uniquely determine the electrical conductivity? In general, not known.
- CDII (Nachman et al since 2002, Seo (2002)) : + two rotation of the object

$$\Rightarrow \mathbf{B} \Rightarrow \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

- Anisotropic case: Bal & Monard (2013), unique determination Hoell-Moradifam-Nachman (2014, within conformal class)

Today: the magnitude $|\mathbf{J}|$ is assumed known inside.
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Complete Electrode Model
(Somersalo-Cheney-Isaacson ’92)

\( \Omega \subset \mathbb{R}^n \) bounded with Lipschitz boundary \( \partial \Omega \),
\( N + 1 \) electrodes: \( e_k \subset \partial \Omega, \ k = 0, ..., N, \)
\( \epsilon \leq \text{Re}\{\sigma}\leq 1/\epsilon, \)
\( \epsilon \leq \text{Re}\{z_k\} \leq 1/\epsilon, \ k = 0, 1, ..., N, \)

\[ \nabla \cdot \sigma \nabla u = 0, \text{ in } \Omega, \]

\[ u + z_k \sigma \frac{\partial u}{\partial \nu} \equiv \text{const} = U_k \text{ on } e_k, \text{ for } k = 0, ..., N, \]

\[ \int_{e_k} \sigma \frac{\partial u}{\partial \nu} \, ds = l_k, \text{ for } k = 0, ..., N, \]

\[ \frac{\partial u}{\partial \nu} = 0, \text{ on } \partial \Omega \setminus \bigcup_{k=0}^{N} e_k, \]
Forward problem (CEM) is well posed

Based on Lax-Milgram lemma:

**Theorem** (Somersalo- Cheney- Isaacson ’92) Provided

\[
\sum_{k=0}^{N} I_k = 0,
\]

there is a unique solution \( \langle u(x), (U_0, \ldots, U_N) \rangle \in H^1(\Omega) \times \mathbb{C}^{N+1} \)
up to a constant.
Normalization

Uniqueness up to a constant:
\[ \langle u(x) + c, (U_0 + c, \ldots, U_N + c) \rangle \] also a solution.

\[ \nabla \cdot \sigma \nabla (u + c) = 0, \quad \text{in } \Omega, \]

\[ (u + c) + z_k \sigma \frac{\partial (u + c)}{\partial \nu} \equiv \text{const} = U_k + c \quad \text{on } e_k, \quad \text{for } k = 0, \ldots, N, \]

\[ \int_{e_k} \sigma \frac{\partial (u + c)}{\partial \nu} ds = I_k, \quad \text{for } k = 0, \ldots, N, \]

\[ \frac{\partial (u + c)}{\partial \nu} = 0, \quad \text{on } \partial \Omega \setminus \bigcup_{k=0}^{N} e_k, \]

**Normalization:** fix a constant by seeking \( \mathbf{U} = (U_0, \ldots, U_N) \) with \( \sum_{k=0}^{N} U_k = 0. \)
New properties in the real valued case

\[ \sigma(x), z_0(x), ..., z_N(x) \in \mathbb{R} \]

\[ \mathbf{U} \in \Pi := \left\{ (U_0, ..., U_N) \in \mathbb{R}^{N+1} : \sum_{k=0}^{N} U_k = 0 \right\} \]

- Maximum Principle for CEM: The maximum and minimum of the voltage potential \( u \) occur on the electrodes.

- A Poicaré Inequality (not necessarily connected with CEM): \( \exists C > 0 \) dependent only on \( \Omega \) and \( e_k \subset \partial \Omega \) such that \( \forall u \in H^1(\Omega) \) and \( \forall \mathbf{U} = (U_0, ..., U_N) \in \Pi : \)

\[
\int_{\Omega} u^2 + \sum_{k=0}^{N} U_k^2 \leq C \left( \int_{\Omega} |\nabla u|^2 \, dx + \sum_{k=0}^{N} \int_{e_k} (u - U_k)^2 \, ds \right)
\]
The Dirichlet principle for the CEM

Consider the functional

$$F_{\sigma}(u, U) := \frac{1}{2} \int_{\partial \Omega} \sigma |\nabla u|^2 \, dx + \frac{1}{2} \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} (u - U_k)^2 \, ds - \sum_{k=0}^{N} I_k U_k.$$ 

Recall $\Omega, \Pi, e_k \subset \partial \Omega, z_k, \text{ for } k = 0, \ldots, N, \sigma, \text{ and}$

$$\sum_{k=0}^{N} I_k = 0 \quad (\ast)$$

**Theorem** (Nachman-T-Veras ’14)

(i) Independently of $(\ast)$:

$$\exists! \ (u, U) = \arg\min_{H^1(\Omega) \times \Pi} F_{\sigma}$$

(ii) If $(\ast)$ holds:

$$(u, U) = \arg\min_{H^1(\Omega) \times \Pi} F_{\sigma} \Leftrightarrow (u, U) \text{ solves CEM}$$
Formulation of an Inverse Problem

Given: $\Omega$, $e_k \subset \partial\Omega$ with $z_k > 0$, and $I_1, \ldots, I_N$, (then $I_0 := -\sum_{k=1}^{N} I_k$), and $|\mathbf{J}| = \sigma |\nabla u|$ inside $\Omega$,

Find $\sigma$. 
Formulation of an Inverse Problem

Given: $\Omega$, $e_k \subset \partial \Omega$ with $z_k > 0$, and $I_k, k = 1, \ldots, N$ (then $l_0 := -\sum_{k=1}^{N} I_k$), and $|J| = \sigma |\nabla u|$ inside,

Find $\sigma$.

Not possible:
$\Omega = (0, 1) \times (0, 1)$,
Top side: $e_1$ with $z_1 > 0$, inject $I_1 = 1$
Bottom side: $e_0$ with $z_0 = z_1 + 1$, extract $l_0 = -1$
Measure the magnitude $|J| \equiv 1$ inside.

Arbitrary $\varphi : [0, 1] \to [\varphi(0), \varphi(1)]$ increasing, Lipschitz with $\varphi(0) + \varphi(1) = 1$.

Then: $u_\varphi(x, y) := \varphi(y)$ voltage for $\sigma_\varphi(x, y) = 1/\varphi'(y)$.
Yet for all such $\varphi$,

$$\sigma_\varphi |\nabla u_\varphi| \equiv 1!$$
Generic non-uniqueness

Let \((u, U) \in H^1(\Omega) \times \Pi\) be the solution of CEM for some \(\sigma\). 
\(\varphi \in Lip(u(\overline{\Omega}))\) be an increasing function of one variable, 
\(\varphi(t) = t + c_k\) whenever \(t \in u(e_k)\), for each \(k = 0, \ldots, N\), and 
constants \(c_k\) satisfying \(\sum_{k=0}^{N} c_k = 0\). Then

\[
u \varphi := \varphi \circ u \tag{1}\]

is a voltage potential for CEM with

\[
\sigma \varphi := \frac{\sigma}{\varphi' \circ u}, \tag{2}
\]

and has the same interior data

\[
\sigma |\nabla u| = \sigma \varphi |\nabla u \varphi|. \]
**Theorem** (Nachman-T-Veras ’14) Recall assumptions on \(\Omega \subset \mathbb{R}^d\) be bounded, connected \(C^{1,\alpha}\), \(e_k \subset \partial\Omega\), \(z_k > 0\), \(I_k\), \(k = 0, \ldots, N\).

Let \((u, U), (v, V) \in H^1(\Omega) \times \Pi\), be the CEM solutions for unknown conductivities \(\sigma, \tilde{\sigma} \in C^{\alpha}(\Omega)\) with

\[
|J| := \sigma|\nabla u| = \tilde{\sigma}|\nabla v| > 0 \text{ a.e. in } \Omega.
\]

Then \(\exists \varphi \in C^1(u(\Omega))\), with \(\varphi'(t) > 0\) a.e. in \(\Omega\), such that

\[
v = \varphi \circ u, \quad \text{in } \Omega,
\]

\[
\tilde{\sigma} = \frac{\sigma}{\varphi' \circ u}, \quad \text{a.e. in } \Omega.
\]

Moreover, for each \(k = 0, \ldots, N\) and \(t \in v(e_k)\),

\[
\varphi(t) = t + (U_k - V_k).
\]
Idea: reduction to a minimization problem

Inverse hybrid problem: Consider

\[ G_{||J||}(v, V) = \int_{\Omega} |J| |\nabla v| dx + \frac{1}{2} \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} (v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k, \]

- solutions of CEM are global minimizers of \( G_{||J||} \) over \( H^1(\Omega) \times \Pi \).
- Geometry of the equipotential sets are uniquely determined! Contrast with Dirichlet

Contrast with functional in the forward model

\[ F_\sigma(v, V) := \frac{1}{2} \int_{\partial\Omega} \sigma|\nabla v|^2 dx + \frac{1}{2} \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} (v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k. \]
Corollaries

- **Phase retrieval** (Nachman-T-Veras’14) Same hypotheses (recall).
  
  \[ |J| = |	ilde{J}| \Rightarrow J = \tilde{J}. \]

- There is uniqueness (and a reconstruction method) from the magnitudes of **two** currents via a local formula (Nachman et al., Lee 2004)

- The $J$-substitution algorithm via magnitudes of **two** currents (Seo et al 2002) converges to the unique solution.
Knowledge of the potential on a boundary curve joining the electrodes restores uniqueness

**Theorem** (Nachman-T-Veras ’14) In addition to the hypotheses of the characterization theorem if

\[ u|_\Gamma = \tilde{u}|_\Gamma + C, \]

for some \( C \), and \( \Gamma \) a curve joining the electrodes, then

\[ u = \tilde{u} + C \text{ in } \overline{\Omega}, \]
\[ \sigma = \tilde{\sigma} \text{ in } \Omega. \]
A minimization algorithm for $G$

$$G_J(v, V) = \int_\Omega |J| |\nabla v| dx + \sum_{k=0}^{N} \int_{e_k} \frac{1}{2z_k} (v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k,$$

Lemma Assume that $v \in H^1(\Omega)$ satisfies

$$\epsilon \leq \frac{a}{|\nabla v|} \leq \frac{1}{\epsilon},$$

for some $\epsilon > 0$, and let $(u, U) \in H^1(\Omega) \times \Pi$ be the unique solution for CEM with $\sigma := a/|\nabla v|$. Then

$$G_a(u, U) \leq G_a(v, V), \quad \text{for all } V \in \Pi.$$  

Moreover, if equality holds then $(u, U) = (v, V)$. 
A minimization algorithm

- With $\sigma_n$ given: Solve CEM for the unique solution $(u_n, U^n)$;
- If
  \[
  \text{essinf} \| \nabla u_n - \nabla u_{n-1} \| > \delta \frac{\epsilon}{\text{essinf} |J|},
  \]
  update
  \[
  \sigma_{n+1} := \min \left\{ \max \left\{ \frac{|J|}{|\nabla u_n|}, \epsilon \right\}, \frac{1}{\epsilon} \right\}
  \]
  and repeat;
- else STOP.

Enough for the phase retrieval:
\[
J \approx |J| \frac{\nabla u_n}{|\nabla u_n|}
\]
Using the voltage on $\Gamma$

Let $n$ be the last iteration and set

$$\sigma_{n+1} := \frac{|J|}{\nabla u_n}.$$

The Characterization Theorem

$$\Rightarrow u(x) \approx f(u_n(x)).$$

Read off the measured data on $\Gamma$ to determine the scaling function $f : u(\Gamma) \rightarrow u_n(\Gamma)$.

Then

$$\sigma(x) \approx \frac{1}{f'(u_n(x))}\sigma_{n+1}(x).$$
Reconstruction results in a numerical experiment

\[ 1 \text{S/m} \leq \sigma \leq 1.8 \text{S/m}, \quad -l_0 = l_1 = 3 \text{mA}, \quad z_0 = z_1 = 8.3 \text{mV} \cdot \text{m}^2 \]

Figure: Exact conductivity (left) vs. reconstructed conductivity (right)
Voltage potential scaling along $\Gamma$

Figure: The scaling function $f$ and its derivative.
Figure: $L^2$-Error: Understood from the stability in the linearized case Kuchment & Steinhauer (2011), Bal (2012)
Some learnings

► in the more realistic CEM, the magnitude of one current density by itself cannot determine an isotropic conductivity
► the magnitude of two currents uniquely determine the conductivity (up to an additive constant)
► in the isotropic case: the phase of the current is uniquely determined from its magnitude (not known in the anisotropic case)
► knowledge of the voltage potential along a curve restores uniqueness
► the method is constructive

Thank you!