Hyperbolic Inverse Problems in Hybrid Imaging

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June 17, 2015

The radiation model (TAT)

- Model by Bal & Ren & Uhlmann & Zhou
- D a bounded region in $\mathbb{R}^3$; k frequency
- $\sigma(x)$ conductivity of medium - $\sigma(x) = 0$ outside D
- Medium probed by an incoming plane wave (or other source)

$$\Delta u + k^2 u + ik\sigma(x)u = 0, \qquad x \in \mathbb{R}^3$$
$$u = e^{ikx\cdot\theta} + u^s$$
u^s satisfies Sommerfeld radiation condition

- Goal: Recover $\sigma(x)$, given $H(x) = \sigma(x)|u(x)|^2$, $\forall\, x \in D$.
- How was $H(x)$ acquired?

The acoustic model (TAT)

- $c(x) > 0$ sound speed, with $c(x) \equiv 1$ for x outside D;
- define $\rho(x) = 1/c(x)^2$
- $v(x,t)$ solution of IVP

$$\begin{aligned} \rho(x) v_{tt} - \Delta v = 0, &\qquad (x,t) \in \mathbb{R}^3 \times [0,\infty) \\ v(x,0) = H(x), \quad v_t(x,0) = 0, &\qquad x \in \mathbb{R}^3 \end{aligned}$$

- Goal: Given $v|_{\partial D \times [0,T]}$, recover $H(\cdot)$ (and $\rho(\cdot)$) .
- Remark: If T large enough then $v|_{\partial D \times [0,T]} = 0$ implies $H = 0$ on D
- Remark: $H(\cdot)$ not arbitrary - solution of Helmholtz equation.

Recovering $H(\cdot)$ with known ρ

- $v(x,t)$ solution of IVP

$$\begin{aligned} \rho(x)v_{tt} - \Delta v &= 0, \qquad (x,t) \in \mathbb{R}^3 \times [0,\infty) \\ v(x,0) = H(x), \quad v_t(x,0) &= 0, \qquad x \in \mathbb{R}^3 \end{aligned}$$

- Goal: Given $\rho(x)$ invert the map $\mathcal{F} : H(\cdot) \to v|_{\partial D \times [0,T]}$.
- A linear problem - formally determined.
- (Stefanov and Uhlmann) $H(x) = (I - K)^{-1}A(data)$ where
  A - a backward in time solver
  K - a forward solver composed with a backward solver
  $\|K\| < 1$, if all rays escape D in time T.

Recovering $\rho(\cdot)$ with known $H(\cdot)$

- $v(x,t)$ solution of IVP

$$\begin{aligned} \rho(x) v_{tt} - \Delta v = 0, \qquad & (x,t) \in \mathbb{R}^3 \times [0,\infty) \\ v(x,0) = H(x), \quad v_t(x,0) = 0, \qquad & x \in \mathbb{R}^3 \end{aligned}$$

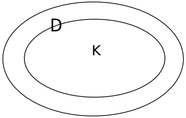
- Goal: Given $H(x)$, invert map $\mathcal{F} : \rho(\cdot) \to v|_{\partial D \times [0,T]}$.
- $\mathcal{F}$ is nonlinear, formally determined.
- Need conditions on H if we hope to recover ρ.
- (Bukhgeim-Klibanov, Imanuvilov-Yamamoto, Stefanov-Uhlmann) If $\mathcal{F}(\rho_1) = \mathcal{F}(\rho_2)$ then $\rho_1 = \rho_2$ provided
  - $(\Delta H) \neq 0$ on $\operatorname{supp} \rho_1 - \rho_2$
  - D is swept out by a family of 'convex' surfaces
  - T is large enough
- Stability also.

Interpretation of ρ recovery result

- $v(x,t)$ solution of IVP

$$\rho(x)v_{tt} - \Delta v = 0, \qquad (x,t) \in \mathbb{R}^3 \times [0,\infty)$$
$$v(x,0) = H(x), \quad v_t(x,0) = 0, \qquad x \in \mathbb{R}^3$$

- Goal: Given $H(x)$, invert map $\mathcal{F} : \rho(\cdot) \to v|_{\partial D \times [0,T]}$.
- Result: If ρ known outside K and ΔH never *zero* in K then $\mathcal{F}$ is injective and $\mathcal{F}^{-1}$ is continuous.



- Questions: Can we expect ΔH to be never zero? Is it possible to weaken this condition? What about reconstruction?

The questions

- $v(x,t)$ solution of IVP

$$\begin{aligned} \rho(x)v_{tt} - \Delta v = 0, \qquad & (x,t) \in \mathbb{R}^3 \times [0,\infty) \\ v(x,0) = H(x), \quad v_t(x,0) = 0, \qquad & x \in \mathbb{R}^3 \end{aligned}$$

- Goal: Given $H(x)$, invert map $\mathcal{F} : \rho(\cdot) \to v|_{\partial D \times [0,T]}$.
- Can we expect ΔH to be never zero?
  Recall $H = \sigma|u|^2$ where $\Delta u + k^2 u + ik\sigma u = 0$ in $\mathbb{R}^3$.
- Can we weaken this condition?
- What about reconstruction? Perhaps in Klibanov's work.
  Also, paper by Baudoin, Buhan, Ervedoza (2013).

Recovering H and ρ

- $v(x,t)$ solution of IVP

$$\begin{aligned} \rho(x)v_{tt} - \Delta v = 0, \qquad & (x,t) \in \mathbb{R}^3 \times [0,\infty) \\ v(x,0) = H(x), \quad v_t(x,0) = 0, \qquad & x \in \mathbb{R}^3 \end{aligned}$$

- Goal: Invert map $\mathcal{F} : (\rho, H) \to v|_{\partial D \times [0,T]}$.
- Known that if $F(\rho, H) = 0$ and T large then $H = 0$ - done.
- Transmission eigenvalues - David Finch, Luc Robbiano
- (Stefanov&Uhlmann) There is no estimate of the form

$$\|\delta H\|_{s_1,D} + \|\delta\rho\|_{s_1,K} \leq C\|\mathcal{F}'(1,H)(\delta\rho,\delta H)\|_{s_2}$$

  regardless of the non-negative s_1, s_2.
- Can δH cannot be varied arbitrarily?
- Is there a way out?

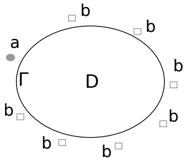
Adding reflectors

- $v(x,t)$ solution of IBVP

$$\begin{aligned} \rho(x)v_{tt} - \Delta v = 0, &\qquad (x,t) \in D \times [0,\infty) \\ v(x,0) = H(x), \quad v_t(x,0) = 0, &\qquad x \in D \\ (\partial_n v)(x,t) + \lambda(x)v_t(x,t) = 0 &\qquad (x,t) \in \partial D \times [0,T]. \end{aligned}$$

- Goal: Invert map $\mathcal{F} : (\rho, H) \to v|_{\partial D \times [0,T]}$.
- If ρ known then recovery of H using results of Acosta & Montalto; Stefanov & Yang.
- What about recovering ρ and H? An example of non-uniqueness when $\lambda = 0$.

Recover ρ using separate acoustic experiments

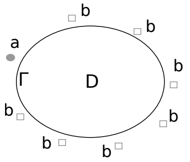


- For any a in Γ (bdry of D) let $v^a(x,t)$ be the solution

$$\begin{aligned} \rho(x)v_{tt} - \Delta v &= \delta(x-a,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ v(x,t) &= 0, \qquad x \in \mathbb{R}^3,\ t < 0. \end{aligned}$$

- Goal: Invert map $\mathcal{F} : \rho \to v^a(b,t)|_{(a,b,t)\in\Gamma\times\Gamma\times[0,T]}$.
- Inversion from hyperbolic D-N map (over-determined problem)
- Inversion via Belishev's boundary control method - not stable.
- Half plane case is Geophysics problem : least square minimization not convex.

Recover ρ using a single acoustic experiment



- For FIXED a in Γ (bdry of D) let $v^a(x,t)$ be the solution

$$\begin{aligned} \rho(x) v_{tt} - \Delta v &= \delta(x-a,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ v(x,t) &= 0, \qquad x \in \mathbb{R}^3,\ t<0. \end{aligned}$$

- Goal: Invert map $\mathcal{F} : \rho \to v^a(b,t)|_{(b,t)\in\Gamma\times[0,T]}$.
- Formally determined inverse problem
- Injectivity of $\mathcal{F}$ is not known.
- [Romanov] $F'(1)$ (linearization around $\rho = 1$) is invertible with bounded inverse.