

Hybrid Methods of Imaging

Guillaume Bal (Columbia University, New York),
Fernando Guevara Vasquez (University of Utah, Salt Lake City),
Leonid Kunyansky (University of Arizona, Tucson)

June 14-19, 2015

1 Overview of the Field

Since its invention in the 1960s, computed tomography (CT) has become an indispensable technique of biomedical imaging. Numerous modalities have been introduced since then, including the traditional X-ray CT scan, SPECT, MRI, Optical-, Ultrasound-, and Electrical Impedance Tomography, with many others being currently developed. All these techniques are used to obtain images of the internal structure of the living tissue in humans or animals. They differ by the underlying physics, by costs (sometimes very significant) and health hazards to the patient (also significant in some modalities). But most importantly, they differ by the biomedical features they can (or cannot) detect. In spite of the wide variety of existing modalities, such tasks as detection of cancerous tumors in soft tissues still present a significant challenge. Thus, the search continues for new, more capable, more sensitive, and more affordable imaging techniques.

Recently, several “hybrid” or “coupled physics” modalities have been introduced. They remain a subject of intensive research activity since then, due to the great promises they hold for medical imaging. By combining two or three different types of waves (or physical fields) these methods overcome limitations of classical tomography techniques and deliver otherwise unavailable, potentially life-saving diagnostic information — at a lesser cost and with less harm to the patient. Among these methods are the Thermoacoustic Tomography, Photo-Acoustic Tomography, Ultrasound Modulated Optical and Impedance Tomographies, Magneto-Acousto-Electric Tomography (MAET) and several other modalities combining magnetic fields with ultrasound scanning of the tissue. Closely related to these methods are so-called combined physics modalities such as Current Density Imaging and Elastography.

As a rule, the images in these modalities are obtained by complex mathematical procedures, rather than through direct acquisition. In most cases, the necessary mathematics involves sophisticated techniques of integral geometry, contemporary theory of partial differential equations, spectral theory, microlocal analysis, numerical analysis, etc. Each time a new modality is introduced, researchers face new mathematical challenges, ranging from the theoretical questions about the existence, uniqueness and stability of the solutions to the equations (representing the images that need to be reconstructed), to the more practical tasks of developing computer algorithms and programs capable of computing the images fast and in high resolution, as required by modern medical practice. These tasks are not trivial; obtaining these results requires collaboration of theoretical and applied mathematicians, as well as an interdisciplinary dialog between the mathematicians and physicists, engineers, and medical practitioners who build and use this state-of-the-art equipment.

2 Recent Developments and Open Problems

2.1 Thermo-, photo-, and opto- acoustic tomography

Thermoacoustic Tomography (TAT) [61, 111] and Photoacoustic (or Optoacoustic) Tomography (PAT) [60, 89, 35] are the most developed of the novel “hybrid” methods of medical imaging. These hybrid (or “coupled physics”) modalities combine different physical types of waves in such a way that the resolution and contrast of the resulting method are much higher than those achievable using only acoustic or electromagnetic measurements. In the case of TAT and PAT, these waves are electromagnetic (EM) waves (infrared laser radiation in PAT and microwaves in TAT) and high-frequency acoustic waves. In these modalities, a short pulse of EM waves irradiates the biological object of interest, thus causing small levels of heating. The resulting thermoelastic expansion generates a pressure wave that starts propagating through the object. The absorbed EM energy and the initial pressure it creates are much higher in the cancerous cells than in healthy tissues (see the discussion of this effect in [110, 112, 111]). Thus, if one could reconstruct the initial pressure, the resulting TAT/PAT tomogram would contain highly useful diagnostic information. The data for such a reconstruction are obtained by measuring time-dependent pressure on a surface completely or partially surrounding the object. Although the initial irradiation is electro-magnetic, the actual reconstruction is based on acoustic measurements. Therefore these methods yield high contrast because of the higher absorption of EM energy by cancerous cells, while also achieving good (sub-millimeter) resolution because of the ultrasound measurements. (The radio frequency EM waves are too long for high-resolution imaging). Thus, TAT/PAT combine the advantages of two types of waves by using them in tandem, while eliminating their individual deficiencies.

TAT and PAT began to be developed as a viable medical imaging technique in the mid 1990s [89, 60]. Some of the mathematical foundations of these imaging modalities were originally developed starting in the 1990s for the purposes of the approximation theory, integral geometry, and sonar and radar (see [3, 73, 2, 63, 48] for references and extensive reviews of the resulting developments). Physical, biological and mathematical aspects of TAT/PAT have been recently reviewed in [63, 2, 48, 47, 90, 107, 109, 111, 112, 99].

2.1.1 The inverse source problem

The first step in solving the inversion problem of TAT/PAT is finding the initial pressure in the tissues originating from the EM excitation; this step is frequently called the inverse source problem.

Explicit inversion formulas The simplest situation to treat is when the speed of sound in the tissues can be approximated by a constant, as is the case, for example, in breast imaging. Under the constant sound speed assumption, a solution of the inverse source problem can be represented by explicit closed-form formulas, for certain simple acquisition surfaces. Such formulas are similar to the well-known filtration/backprojection formula in X-ray tomography; they allow one to compute the initial pressure by evaluating an explicit integro-differential operator at each node of a reconstruction grid. The existence and form of explicit inversion formulas are closely related to the shape of the data acquisition surface. For the simplest case of a planar surface, explicit formulas have been known for several decades [14, 43, 41]. The authors of [113] found a filtration/backprojection formula that is valid for a plane, a 3D sphere and an infinite 3D cylinder. In [46, 44, 68, 85] several different inversion formulas were derived for spherical acquisition surfaces in spaces of various dimensions. In [65] explicit reconstruction formulas were obtained for detectors placed on a surface of a cube (or square, in 2D) or on surfaces of certain other polygons and polyhedra. Several authors [98, 91, 84, 51] recently found explicit inversion formulas for the data measured from surfaces of an ellipse (in 2D) or an ellipsoid (in 3D) surrounding the object. These formulas can be further extended by continuity to an elliptic paraboloid or parabolic cylinder [53, 52]. Certain more complicated polynomial surfaces (including a paraboloid) were considered in [91], although not all of these surfaces are attractive from a practical point of view, as they would require surrounding the object by several layers of detectors. In addition to the explicit formulas, there exist several reconstruction algorithms (for closed acquisition surfaces) based on various series expansions [86, 87, 69, 66, 54]. These techniques sometimes lead to very fast implementations (e.g. [69, 66]); however, their efficient numerical implementation may require certain non-trivial computational skills.

Operator solution In the more general case of variable (but known) speed of sound in the tissues, the solution of the inverse source problem is more complicated. In [1], a general operator inversion formula is obtained, which applies to arbitrary geometry of the observation surface, variable sound speed under a non-trapping condition and functions not necessarily supported inside the observation surface. The formula is written in terms of operator functions of the Dirichlet Laplacian on the domain surrounded by the acquisition surface, and thus it is not easy to apply. However, it also leads to an eigenfunction expansion method that generalizes the series algorithm of [69] to the case of a variable sound speed, which is more tractable computationally.

Time reversal Time reversal was successfully used by multiple authors, both theoretically and as a computational technique, to solve the inverse source problem of TAT/PAT ([46, 58, 103, 57, 97, 104, 56]). This method consists in solving the wave equation backwards in time, within the domain Ω surrounded by the detectors, using the measured data as the Dirichlet boundary conditions on the boundary $\partial\Omega$ of Ω . Theoretical foundations of the method are the simplest in the case of the constant speed of sound in 3D space. Under these assumptions, the Huygens principle guarantees that the pressure within the domain and on its boundary vanishes after a finite time T_0 . Therefore, one can impose zero initial conditions at $t = T \geq T_0$, and solve the wave equation backwards in time in the domain $(0, T) \times \Omega$. Since the wave equation with such initial and boundary conditions admits a unique solution, at the time $t = 0$ the so-computed solution will coincide with the sought initial pressure in the direct problem.

In the 2D case and/or when the speed of sound is not constant and is non-trapping, the pressure vanishes only as $t \rightarrow \infty$. If only a finite amount of data is collected for $t \in [0, T]$ with sufficiently large T , time reversal still yields a good approximation. In order to reconstruct correctly the singularities of the solution, one has to truncate the data smoothly ([58, 57]), or to initialize the solution using the harmonic extension of the boundary data at $t = T$ ([103, 97, 104]). Alternatively, a theoretically exact solution can be obtained in the form of converging Neumann series ([103, 97, 104]), although in practice the first term in the series is close enough (i.e., the error in the first approximation is less than the errors arising from noise in the data, insufficient sampling, etc.).

2.1.2 Quantitative TAT/PAT

One of the active areas of current research is the so-called quantitative PAT (QPAT) [37, 96, 106] which aims to recover, in addition to the initial pressure, optical properties of the tissue (e.g., Grüneisen coefficient) and the fluency of electromagnetic radiation as it propagates through inhomogeneous tissue.

In the setting of quantitative PAT, the initial acoustic pressure reconstructed above takes the form $H(x) = \Gamma(x)\sigma(x)u(x)$, where $\Gamma(x)$ is the Grüneisen coefficient quantifying the photo-acoustic effect (how much acoustic pressure is generated per absorbed photon), $\sigma(x)$ is the optical absorption coefficient, and $u(x)$ is the light intensity reaching the point x . Note that $\Gamma(x)$ and $\sigma(x)$ are constitutive (optical) properties of the biological tissues. Thus reconstructing $\Gamma(x)$ and $\sigma(x)$ could provide information about the health of these biological tissues. In contrast, $u(x)$ is a quantity that depends on the experimental setting, namely how light is injected into the domain and how it propagates in it.

The quantitative step of hybrid inverse problems precisely aims at quantifying which constitutive parameters may be reconstructed and with which stability. This now requires a model for light propagation. Assuming a well-accepted diffusion model, which is accurate for optically sufficiently large objects, then $u(x)$ is the solution to the following elliptic equation:

$$-\nabla \cdot D(x)\nabla u(x) + \sigma(x)u(x) = 0,$$

in a domain $X \subset \mathbb{R}^n$ with, say, Dirichlet conditions $u = f$ on the boundary ∂X . Here $D(x)$ is an additional diffusion coefficient. Recall that we now know $H(x) = \Gamma(x)\sigma(x)u(x)$. What information on the unknown parameters $(D(x), \sigma(x), \Gamma(x))$ may we infer from knowledge of H and the above PDE constraint? Clearly two constraints cannot possibly help us reconstruct four parameters (since $u(x)$ is also unknown). If instead of considering a single experiment with boundary condition f we considered J different experiments with boundary conditions $\{f_j\}_{1 \leq j \leq J}$, we would know $H_j = \Gamma(x)\sigma(x)u_j(x)$, for $1 \leq j \leq J$. What can we reconstruct for different values of J , and with which stability? How do such reconstructions depend on the choice of the boundary conditions $\{f_j\}$?

This kind of questions, for the modality QPAT as well as for many other hybrid inverse problems, was a central theme in the workshop. For QPAT modeled by a diffusion equation, we have a relatively complete theory available to us. The upshot is that independent of the value of J , there is no chance to reconstruct all three parameters $D(x)$, $\sigma(x)$ and $\Gamma(x)$ simultaneously. All one can reconstruct is two *independent* functionals of these three parameters. When one such parameter is known a priori, then the other two can be reconstructed. Moreover, for sufficiently large J , the reconstruction of the parameters may be formulated as an elliptic system of differential or pseudo-differential operators that provide optimal stability estimates, in the sense that we know how many times data need to be differentiated in the reconstruction of said parameters. For references on such results and generalizations of what is described above, see for instance [25, 30, 28, 40, 37, 64].

In the setting of quantitative TAT, the propagating radiation is much lower frequency and has to be modeled by the Maxwell equations instead of the above diffusion model. The initial pressure is then given by $H(x) = \Gamma(x)\gamma(x)|E|^2(x)$, where Γ is still the Grüneisen coefficient, γ is the electric conductivity and E is the electric field. For references on the results that have been obtained for this problem, see e.g. [13, 29, 33].

2.1.3 Open problems in TAT/PAT/QPAT

Problem with incomplete data In most practical situations the object of interest cannot be completely surrounded by the acoustic detectors. (Perhaps, the only exception is imaging of small animals.) Most of the theoretical and algorithmic results obtained for the inverse source problem in TAT/PAT are based on the assumption of the complete acquisition (from a closed acquisition surface). Theoretical analysis shows (see, e.g., [63]) that stable solution of such an inverse problem with incomplete data is only possible if the object and the acquisition surface satisfy the “visibility condition”: every bi-characteristic of the wave equation originating at each point of the domain of interest must reach the detectors in finite time. The existing constructive approaches to treat the problem with incomplete data are either computational in nature [70], or are parametric-based [94, 95, 92], i.e. aim to reconstruct accurately only the jumps of the function but not the quantitatively correct values. Numerical experimentation has shown [103, 97] that a satisfactory solution can be obtained by an iterative algorithm (Neumann series), in the case when the visibility condition is satisfied. However, in the case of open space propagation of acoustic waves convergence of such iterations has not been proven yet

Reflecting boundaries Practically all existing theory of TAT/PAT is based on the assumption that acoustic waves propagate in free space, and that reflections from detectors and the walls of the water tank can be either neglected or gated out. In this case, acoustic pressure within the object vanishes quite fast and the inverse problem of TAT/PAT can be solved by time reversal. Time reversal yields a theoretically exact reconstruction if the speed of sound is constant or if it satisfies the so-called non-trapping condition (see [1, 58, 103, 97]). This method can be implemented for a general closed acquisition surface and known speed of sound using finite differences [58, 97]; it can also be realized (for simple domains) using the method of separation of variables [69], or, for certain geometries, it can be replaced by equivalent explicit backprojection formulas [65]. Other reconstruction algorithms, although not related directly to time reversal, also require that the pressure vanishes sufficiently fast [63].

However, free space propagation cannot always be used as a valid model. For example, one of the most advanced PAT acquisition schemes (developed by researchers from the University College London [42]) uses optically scanned planar glass surfaces for the detection of acoustic signals. Such surfaces act as (almost) perfect acoustic mirrors. If the object is surrounded by such reflecting detectors (or by a combination of detectors and acoustic mirrors), wave propagation occurs in a resonant cavity. It involves multiple reflections of waves from the walls, and, if the dissipation of waves is neglected, the acoustic oscillations do not die out. Traditional time reversal and other existing techniques are not applicable in this case; new reconstruction algorithms need to be developed for TAT/PAT within resonant cavities. There are few works on this problem [38, 39, 67, 55]. Some of these techniques (e.g. [67]) work only in special geometries and with constant speed of sound. Others ([55]) are more general, but the analysis requires spectral information not easily available aside of several simple cases. This problem requires further investigation.

2.2 Magnetic Resonance Elastography

Magnetic Resonance Elastography (MRE) [80, 74, 101] is a non-invasive medical imaging technique that measures the mechanical properties (stiffness) of soft tissues by introducing shear waves and imaging their propagation using MRI. Pathological tissues are often stiffer than the surrounding normal tissue. For instance, malignant breast tumors are much harder than healthy fibro-glandular tissue. This characteristic has been used by physicians for screening and diagnosis of many diseases through palpation. MRE calculates the mechanical parameter as elicited by palpation, in a non-invasive and objective way.

In magnetic resonance elastography, shear waves are generated by an electro-mechanical transducer on the surface of the skin. (Shear waves are mechanical waves in elastic tissue in which oscillation occurs in the direction normal to the propagation direction). The propagation of the shear waves is then registered (as a function of time and space) by the MRI scan, with the scan modulated by the same frequency as the frequency of the shear waves. This encodes the amplitude of the shear wave in the tissue in the phase of the MRI image. The inverse problem arising in this modality is to reconstruct a quantitative measure of tissue stiffness (an elastogram) from the MRI images.

MRE is currently being investigated as a diagnostic tool for a multitude of diseases affecting tissue stiffness. This modality is being clinically used for the assessment of hepatic fibrosis, since it is well known that the liver stiffness increases with the progression of this disease. MRE is also being utilized for monitoring treatment efficacy of fibrosis and related diseases.

The micro-MRE (magnetic resonance elastography) and PVS (pendulum-type viscoelastic spectrometer) are measurement devices to measure the viscoelasticity of a medium. More precisely, micro-MRE provides an interior measurement of time harmonic waves inside a medium. PVS provides a boundary measurement, obtained by a laser, of the displacement of a specimen under time harmonic torsion or bending generated by Lorentz force (i.e. subjecting the specimen to an oscillating magnetic field). The frequency range of measurement for PVS is much larger than that of for micro-MRE, and the sample size for micro-MRE is much larger than that of for PVS. While MRE has been successful for diagnosing liver diseases, the study of micro-MRE is to provide a benchmark test for MRE.

Recent advances in MRE and micro-MRE were the subject of several talks at the workshop; see also below for additional details on the second, quantitative, step of elastography, which is shared by both magnetic resonance elastography as described above and ultrasound elastography [24, 50, 88].

2.3 Other hybrid modalities

Several other hybrid modalities beyond Photo-acoustic tomography and Elastography have been analyzed in the mathematical literature and have been discussed during the workshop. While such modalities involve a first modeling step that we are not going to describe here, their mathematical analysis primarily involves solving a quantitative inverse problem from knowledge of internal data, as was the case for the quantitative PAT and TAT problems described in section 2.1.2.

These modalities are all described by a partial differential constraint involving unknown coefficients and an internal constraint, also possibly involving unknown coefficients. Here is a partial list of examples. In the quantitative step of Elastography, one seeks to reconstruct elastic properties of biological tissues from knowledge of internal displacements, which were provided by MRE or Ultrasound Elastography as mentioned above. In a scalar model for such displacements, one aims to understand what may be reconstructed from (a, b, c) in an elliptic equation of the form

$$(-a_{ij}\partial_i\partial_j + b_i\partial_i + c)u_k = 0, \quad X, \quad u_k = f_k, \quad \partial X$$

(with Einstein convention of summation of repeated indices) from knowledge of u_k in X for $1 \leq k \leq K$. This problem, as well as generalizations to time dependent problems and systems of elasticity and applications to other fields such as hydraulics, are analyzed in, e.g., [7, 19, 31, 32, 34, 72, 75, 76].

In the application of Magnetic Resonance Electrical Impedance Tomography (MREIT), also known as Current Density Impedance Imaging (CDII), one seeks to reconstruct the (possibly tensor-valued) conductivity γ from knowledge of the current $J = \gamma \nabla u$ or its norm $|J| = \gamma |\nabla u|$, where u solves the elliptic model $\nabla \cdot \gamma \nabla u = 0$ on X with appropriate boundary conditions. This problem has been extensively studied recently, with a list of references that includes [22, 59, 83, 82, 81, 100].

In the modalities called Ultrasound Modulation (Electrical or Optical) Tomography or acousto-optics tomography, the electrical or optical properties of biological tissues are modified by ultrasound modulation. This results in problems with known internal functionals of the form $H = \gamma \nabla u \cdot \nabla u + \eta \sigma u^2$ for known constant η , where u is a solution of an elliptic equation of the form $-\nabla \cdot \gamma \nabla u + \sigma u = 0$ on a domain X with appropriate boundary conditions. The modeling and analysis of this mathematically difficult problem is addressed in a series of papers such as [11, 12, 17, 20, 23, 26, 27, 36, 49, 77, 78, 79].

Each of these problems display specific features that are not shared by other inverse problems. There are, however, common themes that most of these hybrid inverse problems share, including a modeling as a system of pseudo-differential or differential equations. General frameworks to analyze these hybrid inverse problems were developed in, e.g., [18, 64].

Also, in most hybrid inverse problems, specific qualitative properties of solutions to elliptic equations need to be satisfied, for instance that there be no critical points, or that the gradients of n solutions form a basis in \mathbb{R}^n at each point in a domain $X \subset \mathbb{R}^n$. A great variety of results have been obtained in this direction, while many more interesting problems remain open. Several recent results were discussed in the workshop; see below. For a partial list of references on such results, the reader may consult [4, 5, 6, 8, 9, 16, 21, 30].

Additional information about hybrid inverse problems may also be found in the following review papers and books [10, 15, 16, 93, 99].

3 Presentation Highlights

3.1 Thermo- and photo- acoustic tomography

One of the first talks of the conference was by **Alexander Oraevsky** (TomoWave Laboratories, Inc.) on of the founders of the Optoacoustic tomography (OAT, also known as PAT). The speaker concentrated on advances made by this modality in two decades after its invention, and on present challenges arising in OAT. Currently, PAT is entering the real world of clinical applications, with diagnostic imaging of breast cancer being the first major market niche for this technology where existing modalities have apparent drawbacks. The main value of OAT is in its potential capability to provide functional and molecular information based on quantitatively accurate display of the optical absorption coefficient. However, quantitatively accurate OAT has not been demonstrated yet. The speaker emphasized the need for a full view three-dimensional tomography system that acquires complete set of forward data and uses rigorous solutions for inverse problem of image reconstruction have the potential for success in the breast cancer diagnostics. As a step in this direction the speaker and collaborators combined laser optoacoustics and ultrasound tomography systems in a single device. The optoacoustic sub-system provides images based on distribution of molecular chromophores in the body, while the ultrasound sub-system provides anatomical images of tissue structures and can also provide the speed of sound and acoustic attenuation images, which can be used for iterative reconstruction of more accurate optoacoustic images.

The topic of practical and computational challenges arising in practical applications of hybrid modalities was continued by **Mark A. Anastasio** (Washington University). The speaker has reviewed recent advancements in practical image reconstruction approaches for Photoacoustic Computed Tomography (PACT), including physics-based models of the measurement process and associated optimization-based inversion methods for reconstructing images from limited data sets in acoustically heterogeneous media. He also discussed applications of PACT to transcranial brain imaging and breast cancer detection.

An overview of several mathematical theories applying to photo-acoustic and thermo-acoustic tomography was presented in **Rakesh's** lecture, including those with an integral geometric flavor [46, 48] as well as those based on the underlying wave equation for sound propagation [45, 103, 105].

Two talks by **Linh Nguyen** (University of Idaho) and by **Eric Todd Quinto** (Tufts University) addressed the problem of incomplete data in TAT and PAT. One of the frequently used practical approaches to this problem is to simply backproject (or backpropagate) the measured data while replacing the missing part of the data by zeroes. This leads to loss of certain features in the reconstructed image and sometimes generates spurious features, or artifacts. In his talk, L. Nguyen considered this problem in terms of the inversion of the spherical means Radon transform with incomplete data, and analyzed geometry and strength of the artifacts in the reconstruction. Continuing the topic, Todd Quinto presented a general paradigm to classify added artifacts in limited data tomography, developed jointly with J. Friel that explains the locations and

properties of added artifacts that appear in limited angle tomography. The speaker used microlocal analysis to understand the effect of data restriction, and provided reconstructions from real and simulated data for X-ray CT, photoacoustic tomography, and the circular transform.

Talks by **Yang Yang** (Purdue University) and **Sebastian Acosta** (Baylor College of Medicine) addressed another important open problem of TAT/PAT — reconstruction in the presence of reflecting boundaries. Y. Yang presented a study of the mathematical model of thermoacoustic tomography in bounded domains with perfect reflecting boundary conditions. The speaker presented an averaged sharp time reversal algorithm (developed jointly with P. Stefanov) which solves the problem with an exponentially converging Neumann series. The presented numerical reconstruction was implemented in both the full boundary and partial boundary data cases. Similarly, S. Acosta considered the problem of photoacoustic and thermoacoustic tomography in the presence of physical boundaries such as reflectors or interfaces, which reflect some wave energy back into the domain. The difference with the previous talk was that non-perfectly reflecting boundaries were considered. The resulting inverse problem was related with a statement in boundary observability and stabilization of waves. The speaker presented uniqueness and stability of the inverse problem and proposed two different reconstruction methods. It was shown that in both cases, if well-known geometrical conditions were satisfied, the inverse problem can be solved under the assumption of variable wave speed and in the case of measurements given on a subset of the boundary.

L. Kunyansky (University of Arizona) presented a talk on inversion formulas for the spherical means transform with centers lying on the boundaries of in corner-like domains, such as a quadrant in 2D or an octant in 3D. The presented formulas allow one to reduce the problem of inversion of the spherical transform to the similar problem formulated in term of the classical Radon transform. Methods and algorithms for solving the latter problems are well known.

As one can see in the previous talk outlines, integral geometry techniques are an essential ingredient in the first, qualitative step of photo-acoustic tomography. Integral geometry of course finds many applications in imaging sciences. This includes the imaging of materials, and at the forefront of current research, the reconstruction of anisotropic coefficients. **Victor Palamodov** presented recent results on the evaluation of the residual elastic strain in structural material that requires imaging a six-component tensor quantity ε in three dimensions by combining two imaging modalities. He first devised a method of reconstruction of small residual strain fields in a body based on data of diffraction pattern under penetrated X-ray or neutron radiation. The mathematical model is the longitudinal (axial) line transform of the tensor ε . These data are only sufficient for determination of the Saint-Venant tensor $V\varepsilon$ which is a 2×2 - symmetric-skewsymmetric tensor field. A complete reconstruction of a strain field by this method is impossible, since the Saint-Venant tensor vanishes for any potential strain field.

As a second imaging modality, the method of polarization tomography is based on measurements of transformation of the polarization ellipse of the penetrating light through a weakly optically anisotropic material. The mathematical model is the line integral $T\varepsilon$ of the traceless normal (truncated transverse) part of the stress field ε . A simple method of reconstruction of the displacement form from Tuy-complete data of $T\varepsilon$ for any tensor whose axial line integrals vanish. As Victor Palamodov showed us, both methods can be combined for complete reconstruction of an arbitrary small strain tensor e from data of axial and traceless normal ray integrals of ε .

3.2 Quantitative Photo-acoustic Imaging (QPAT)

As indicated in section 2.1.2, QPAT aims at reconstructing optical coefficients from knowledge of an initial pressure pressure of the form $\Gamma(x)\sigma(x)u(x)$. Three talks, by Giovanni Alberti, Kui Ren, and Otmar Scherzer, analyzed various aspects of QPAT.

In his talk, **Kui Ren** considered the setting where $u(x)$ is modeled as the angular average of a phase space photon density $v(x, \theta)$, which then solves a linear Boltzmann equation. Kui Ren presented several new theoretical results on this problem, as well as applications to the related problem of fluorescence photo-acoustic tomography.

In the previous talk, as well as in many past works on QPAT, a precise model for $u(x)$ is necessary. **Giovanni Alberti's** presentation focuses on situations where such model may not be known precisely, as is arguably the case in photo-acoustics in some settings. Let us write $f = \log(\Gamma\sigma)$ and $g_i = \log u_i$. The main focus of his talk was the reconstruction of the signals f and g_i , $i = 1, \dots, N$, from the knowledge of their

sums $h_i = f + g_i$, under the assumption that f and the g_i s can be sparsely represented with respect to two different dictionaries A_f and A_g . This generalises the well-known “morphological component analysis” to a multi-measurement setting. The main result states that f and the g_i s can be uniquely and stably reconstructed by finding sparse representations of h_i for every i with respect to the concatenated dictionary $[A_f, A_g]$, provided that enough incoherent measurements g_i s are available. The incoherence is measured in terms of their mutual disjoint sparsity. Giovanni Alberti showed us how to apply such a disjoint sparsity approach in quantitative photoacoustic tomography, including in the case when the Grüneisen parameter, the optical absorption and the diffusion coefficient are all unknown.

In his presentation based on joint work with Elena Beretta (Milan), Markus Grasmair (Trondheim), Monika Muskieta (Wroclaw), and Wolf Naetar (Vienna), **Otmar Scherzer** further considered the reconstruction of diffusion, absorption, and Grüneisen parameters in QPAT. Recall from [28] that the three parameters cannot uniquely be reconstructed without prior information. Otmar Scherzer showed us that when one assumes piecewise constant diffusion, scattering, and Grüneisen parameters, respectively, then this problem can be decomposed into edge detection problem for the fluence ($u(x)$) and its derivatives and a parameter selection process based on the jump relations of the diffusion equation. Novel edge detection algorithms tuned to these problems have been presented.

3.3 Magnetic Resonance Elastography

Mayo Clinic is one of the birthplaces of Magnetic Resonance Elastography (MRE), and, currently, one of the leading institutions in the development of this modality. It was represented at the workshop by **Armando Manduca** who gave a talk “*Magnetic Resonance Elastography: A Signal Processing Perspective*”. The speaker concentrated on strategies of improving the performance of MRE. He presented a statistical signal processing framework for steady-state MRE that enables rigorous characterization of the accuracy, precision, and uniqueness of harmonic motion information estimated from the raw data collected by an MRI scanner. After deriving and demonstrating the utility of this framework, the speaker discussed statistical strategies for optimally estimating MRE harmonic information directly from raw MRI data, overviewed several unique mathematical aspects of this problem, and presented a robust computational strategy for solving this problem.

Joyce R. McLaughlin (Rensselaer Polytechnic Institute) gave a talk on issues related to stability and statistics in shear stiffness imaging. She considered two different setups common in MRE. In the first measurement technique, the tissue is excited with a time harmonic oscillation and then sequences of magnetic resonance data are taken and processed to produce a movie of the oscillating tissue within the body. For this experiment the speaker presented stability results for a single elastic vector movie. In the second experiment one pulse or a sequence of pulses are imparted by focusing ultrasound; a wave with a front propagates away from the pulse position. The arrival time of one component of the wave is calculated from the movie created from a sequence of ultrasound data sets. (Technically, this modality is not MRE, but a closely related sonoelastography). The speaker investigated statistical properties of the noise in the image when using the direct algorithm, showing that even though the variance is infinite there are some favorable statistical properties.

Gen Nakamura (Hokkaido University) presented his latest results on micro-MRE and PVS. The speaker highlighted the importance of having a good model equation for the measurements and good inversion scheme to recover viscoelasticity from the measurements, in order to get some approximate true value of viscoelasticity of a medium. He introduced the background and motivation for the latest study, and presented the data analysis of these two rheological measurement devices.

3.4 Current Density Imaging, Acousto-Optic Imaging, Seismic Imaging, and Hybrid Inverse Problems features

Many other modalities were discussed at the workshop.

Amir Moradifam and Alexandru Tamasan gave two lectures on their recent work in Current Density Impedance Imaging (CDII), where the objective is to reconstruct γ from knowledge of $J = |\gamma \nabla u|$ where u is the solution to the elliptic equation $\nabla \cdot \gamma \nabla u = 0$ with appropriate boundary conditions.

Alexandru Tamasan (U of Central Florida) considered the inverse problem with boundary conditions given by the Complete Electrode Model introduced by Somersalo, Cheney and Isaacson [102]. As in the

setting with Dirichlet boundary conditions [83], the inverse problem is modeled as a weighted minimum gradient problem. The main result obtained by Tamasan, joint with A. Nachman and J. Veras, is that knowledge at the boundary of the electrodes and their average input currents allows us to obtain the level sets of u but not their values. Additional knowledge of u on an appropriate line at the boundary then uniquely characterizes u and σ .

Amir Moradifam (U of California, Riverside), in a joint work with Robert L. Jerrard and Adrian Nachman, recast the CDII problem as an example of the general least gradient problem

$$\inf_{u \in BV_f(\Omega)} \int_{\Omega} \varphi(x, Du),$$

where $f : \partial\Omega \rightarrow \mathbb{R}$ is continuous, $BV_f(\Omega) := \{v \in BV(\Omega) : v|_{\partial\Omega} = f\}$, and $\varphi(x, \xi)$ is a function that, among other properties, is convex and homogeneous of degree 1 with respect to the ξ variable. The lecture covered existence, uniqueness, and comparison theorems for such minimizers. In particular, it was shown that if $a \in C^{1,1}(\Omega)$ was bounded away from zero, then minimizers of the weighted least gradient problem $\inf_{u \in BV_f} \int_{\Omega} a|Du|$ were unique in $BV_f(\Omega)$. Counterexamples showed that the regularity assumption $a \in C^{1,1}$ was sharp.

While hybrid inverse problems often find applications in medical imaging, some are concerned with seismic imaging. In fluid-saturated porous media, electromagnetic fields couple with seismic waves through the electroseismic conversion. In his talk, **Jie Chen** (Purdue University) presented recent results obtained with M. De Hoop on the mathematical analysis of the electroseismic conversion, and in particular showed us a stability result of recovering the seismic sources from the boundary seismic measurements followed by the inversion of a system of Maxwell's equations with internal data.

Alexander Mamonov (U of Houston) tackled the seismic imaging problem directly from boundary seismic data. In a joint work with Vladimir Druskin and Mikhail Zaslavsky, they introduced a novel nonlinear seismic imaging method based on model order reduction. The reduced order model (ROM) is an orthogonal projection of the wave equation propagator operator on the subspace of the snapshots of the solutions of the wave equation, which allows for the removal of multiple reflection artifacts and also enables us to estimate the magnitude of the reflectors similarly to the true amplitude migration algorithms. Numerical results for the standard Marmousi model and a synthetic high contrast hydraulic fracture example showed the usefulness of the new approach.

Acousto-optics was described earlier in these pages as an ultrasound modulation of the optical properties of a domain of interest. **John Schotland** (U of Michigan) presented a novel method to reconstruct the optical properties of a highly-scattering medium from acousto-optic measurements. The method is based on the solution to an inverse problem for the radiative transport equation with internal data. A stability estimate and a direct reconstruction procedure were described in this talk based on a joint work with G. Bal and F. Chen.

The essential reason for the ultrasound modulation in acousto-optics as well as ultrasound modulated electrical impedance tomography (EIT) is that these inverse problems are severely ill-posed in the absence of such a modulation. Their analysis nonetheless remains an essential aspect of theoretical inverse problems. Moreover, in spite of their limited resolution, such ill-posed inverse problems find important applications in medical imaging. Two talks were devoted to the analysis and the applications of variations of the standard Calderón problem [108].

In his presentation, **David Isaacson** (Rensselaer Polytechnic Institute) introduced a method to image the ventilation perfusion (VQ) ratio using EIT, which is the ratio of the volume of air entering a region of the lungs per breath divided by the volume of blood entering the same region per heart beat. This simple, macroscopic, quantity finds extremely useful applications in medical imaging. After providing means to define, measure, and image an approximation to this VQ ratio using electrical impedance imaging, David Isaacson showed us the relevance of such a test based on images, movies, and data from human subject studies obtained using the RPI and GE electrical impedance imaging systems.

The analysis of stability estimates for the variants of the Calderón problem remains a very active area of research. **Kaloyan Marinov** (Technical University of Denmark) presented recent joint results with Pedro Caro on the inverse boundary-value problems in an infinite slab with partial data. Generalizing results obtained by Li and Uhlmann [71] for the Schrödinger equation and by Krupchyk, Lassas and Uhlmann [62] for the magnetic Schrödinger equation, the obtained a log-log stability estimate for such inverse problems. Two settings of boundary measurements were considered: in the first inverse problem, the corresponding Dirichlet

and Neumann data are known on different boundary hyperplanes of the slab; in the second inverse problem, they are known on the same boundary hyperplane of the slab.

Let us finally comment on the lecture given by **Yves Capdeboscq** (U of Oxford). We have mentioned earlier that many hybrid inverse problems required specific qualitative properties of solutions of elliptic equations. One such property is that the gradients of n solutions form a basis in \mathbb{R}^n at each point inside a domain of interest X . It is relatively straightforward to construct such solutions (harmonic polynomials will do) for solutions of $\nabla \cdot \gamma \nabla u = 0$ when γ is sufficiently close to identity. Yves Capdeboscq presented a striking result essentially showing that in dimension $n \geq 3$, a choice of boundary conditions that works for a class of γ , in the sense that the n gradients are linearly independent inside X , does not work for another appropriate class of highly oscillatory conductivities γ . In other words, it may be the case that for certain hybrid inverse problems, the number of measurements that need to be performed depends on the structure of the unknown coefficients in order to afford reconstruction procedures with optimal stability. These results in dimension $n \geq 3$ are in sharp contrast to the two-dimensional setting; see [9].

4 Scientific Progress Made & Outcome of the Meeting

Hybrid inverse problems have emerged as a recent subfield of inverse problems theory to model, analyze, and compute, several coupled-physics imaging modalities recently introduced by biomedical engineers. This area of research benefited from outstanding amount of work done by the applied mathematics community, as these pages hopefully convinced the reader. Its analysis involves many classical methods of inverse boundary value problems, such as integral geometric techniques, and inverse wave propagation problems that well studied in applications to medical and geophysical imaging.

This field also involves areas of mathematics that were not before of central interest in inverse problems. One main difference can be seen in the quantitative, second, step, of hybrid inverse problems, where reconstructions are performed from knowledge of internal functionals, which, as useful as they are, do not directly provide quantities of interest such as the elastic, electrical, or optical properties of biological tissues. Another new field of study came from the realization that hybrid inverse problems could optimally be solved provided that solutions of partial differential equations satisfied specific qualitative properties, such as not having any critical points, having gradients or Hessians of solutions that are of maximal rank, or light cones associated to some solutions with trivial intersection [18]. The aforementioned talk by Capdeboscq offered very interesting results on the problem of maximal rank of gradients of elliptic solutions.

More broadly, the field of hybrid inverse problems provides a wonderful platform for mathematicians and engineers to collaborate and foster a very promising area of research in biomedical imaging. The Banff International Research Station is an ideal environment for necessary discussions involving researchers with very different backgrounds to take place.

While it is early to assess the full impact of the meeting, here are a few points we can make on the scientific outcome of the meeting and its scientific value.

The two most successful hybrid imaging modalities at the moment are elastography and photo- and thermo- acoustic imaging. These modalities were highly represented at the meeting, both from the engineering and mathematical points of view. Very lively discussions involving, among other researchers, Giovanni Alberti, Mark Anastasio, Guillaume Bal, Yves Capdeboscq, Amir Moradifam, and Alexandre Oraevsky took place to devise strategies in quantitative photo-acoustic tomography that would be both mathematically sound and feasible from an engineering perspective.

Elastography was also very well represented, mostly with researchers working on magnetic resonance tomography. Armandu Manduca, Joyce McLaughlin, and Gen Nakamura, will continue their long-standing collaborations to push the limits of this imaging modality, which currently performs extremely well in applications where resolution is not too demanding, and which is currently being developed to be used in more challenging applications such as brain imaging, that require much finer resolution.

From a mathematical perspective, the search for new methods that allow us to analyze the qualitative properties of PDE solutions remains very active. Giovanni Alberti and Guillaume Bal are currently collaborating on a method to prove that critical points of solutions to elliptic equations always exist for well chosen conductivities in dimension $n \geq 3$ independent of the imposed Dirichlet boundary condition. Several new results in this direction are expected to be announced in a near future, some of which being at least partially

attributed to the discussions that took place during the meeting.

A set of lively conversations between Lin Nguyen and Leonid Kunyansky that took place during the workshop will result in a joint paper, currently being prepared for publication.

This, very partial, list of outcomes of the meeting is but one indication of the feeling shared by most participants, that the meeting had been an extremely useful one. A similar meeting "Mathematical Methods in Emerging Modalities of Medical Imaging" took place at BIRS in 2009. The list of references appended to the present report shows that an incredible amount of research in this area has been conducted since then. Some of that research can be directly linked to interactions among mathematicians and engineers that took place during that meeting. We are confident that the 2015 meeting will prove to be equally successful.

References

- [1] M. Agranovsky and P. Kuchment. Uniqueness of reconstruction and an inversion procedure for thermoacoustic and photoacoustic tomography with variable sound speed. *Inverse Problems*, 23(5):2089–2102, 2007.
- [2] M. Agranovsky, P. Kuchment, and L. Kunyansky. On reconstruction formulas and algorithms for the thermoacoustic and photoacoustic tomography. in *Photoacoustic imaging and spectroscopy L. H. Wang (Editor)*, CRC Press, 2009.
- [3] M. L. Agranovsky and E. T. Quinto. Injectivity sets for the Radon transform over circles and complete systems of radial functions. *J. Funct. Anal.*, 139(2):383–414, 1996.
- [4] G. S. Alberti. On multiple frequency power density measurements. *Inverse Probl.*, 29:115007, 2013.
- [5] G. S. Alberti. Enforcing local non-zero constraints in PDEs and applications to hybrid imaging problems. *Comm. PDE*, 40:1855–1883, 2015.
- [6] G. S. Alberti and Y. Capdeboscq. On local non-zero constraints in PDE with analytic coefficients. *Contemp. Math. (to appear)*, 2015.
- [7] G. Alessandrini. An identification problem for an elliptic equation in two variables. *Ann. Mat. Pura Appl.*, 145:265–296, 1986.
- [8] G. Alessandrini, M. Di Cristo, E. Francini, and S. Vessella. Stability for quantitative photoacoustic tomography with well chosen illuminations. *submitted*, 2015.
- [9] G. Alessandrini and V. Nesi. Univalent e^σ -harmonic mappings. *Arch. Rat. Mech. Anal.*, 158:155–171, 2001.
- [10] H. Ammari. *An Introduction to Mathematics of Emerging Biomedical Imaging*, volume 62 of *Mathematics and Applications*. Springer, New York, 2008.
- [11] H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter, and M. Fink. Electrical impedance tomography by elastic deformation. *SIAM J. Appl. Math.*, 68:1557–1573, 2008.
- [12] H. Ammari, Y. Capdeboscq, F. De Gournay, A. Rozanova-Pierrat, and F. Triki. Microwave imaging by elastic perturbation. *SIAM J. Appl. Math.*, 71(6):2112–2130, 2011.
- [13] H. Ammari, J. Garnier, W. Jing, and L. Nguyen. Quantitative thermo-acoustic imaging: An exact reconstruction formula. *J. Differential Equations*, 254:1375–1395, 2013.
- [14] L.-E. Andersson. On the determination of a function from spherical averages. *SIAM J. Math. Anal.*, 19(1):214–232, 1988.
- [15] S. R. Arridge and O. Scherzer. Imaging from coupled physics. *Inverse Problems*, 28:080201, 2012.
- [16] G. Bal. Hybrid inverse problems and internal functionals. In G. Uhlmann, editor, *Inside Out II*. MSRI Publications, Cambridge University Press, Cambridge, UK, 2012.

- [17] G. Bal. Cauchy problem for Ultrasound modulated EIT. *Analysis & PDE*, 6(4):751–775, 2013.
- [18] G. Bal. Hybrid Inverse Problems and Redundant Systems of Partial Differential Equations. In P. Stefanov, A. Vasy, and M. Zworski, editors, *Inverse Problems and Applications*, volume 619 of *Contemporary Mathematics*, pages 15–48. AMS, 2014.
- [19] G. Bal, C. Bellis, S. Imperiale, and F. Monard. Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements. *Inverse Problems*, 30:125004, 2014.
- [20] G. Bal, E. Bonnetier, F. Monard, and F. Triki. Inverse diffusion from knowledge of power densities. *Inverse Problems and Imaging*, 7(2):353–375, 2013.
- [21] G. Bal and M. Courdurier. Boundary control of elliptic solutions to enforce local constraints. *J. Differential Equations*, 255(6):1357–1381, 2013.
- [22] G. Bal, C. Guo, and F. Monard. Inverse anisotropic conductivity from internal current densities. *Inverse Problems*, 30(2):025001, 2014.
- [23] G. Bal, C. Guo, and F. Monard. Linearized internal functionals for anisotropic conductivities. *Inverse Problems and Imaging*, 8(1):1–22, 2014.
- [24] G. Bal and S. Imperiale. Displacement reconstructions in ultrasound elastography. *SIAM J. Imaging Sci.*, 8(2):1070–1089, 2015.
- [25] G. Bal, A. Jollivet, and V. Jugnon. Inverse transport theory of Photoacoustics. *Inverse Problems*, 26:025011, 2010.
- [26] G. Bal and S. Moskow. Local inversions in ultrasound modulated optical tomography. *Inverse Problems*, 30(2):025005, 2014.
- [27] G. Bal, W. Naetar, O. Scherzer, and J. Schotland. Numerical inversion of the power density operator. *J. Ill-posed Inverse Problems*, 21:265280, 2013.
- [28] G. Bal and K. Ren. Multi-source quantitative PAT in diffusive regime. *Inverse Problems*, 27(7):075003, 2011.
- [29] G. Bal, K. Ren, G. Uhlmann, and T. Zhou. Quantitative thermo-acoustics and related problems. *Inverse Problems*, 27(5):055007, 2011.
- [30] G. Bal and G. Uhlmann. Inverse diffusion theory for photoacoustics. *Inverse Problems*, 26(8):085010, 2010.
- [31] G. Bal and G. Uhlmann. Reconstructions for some coupled-physics inverse problems. *Applied Math. Letters*, 25(7):1030–1033, 2012.
- [32] G. Bal and G. Uhlmann. Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions. *Commun. Pure Appl. Math.*, 66:1629–1652, 2013.
- [33] G. Bal and T. Zhou. Hybrid inverse problem for a system of Maxwell’s equations. *Inverse Problems*, 30:055013, 2014.
- [34] P. Bardsley and F. Guevara Vasquez. Restarted inverse Born series for the Schrödinger problem with discrete internal measurements. *Inverse Problems*, 30:045014, 2014.
- [35] P. Beard. Biomedical photoacoustic imaging. *Interface Focus*, 2011.
- [36] Y. Capdeboscq, J. Fehrenbach, F. de Gournay, and O. Kavian. Imaging by modification: numerical reconstruction of local conductivities from corresponding power density measurements. *SIAM J. Imaging Sciences*, 2:1003–1030, 2009.
- [37] B. Cox, J. G. Laufer, S. R. Arridge, and P. C. Beard. Quantitative spectroscopic photoacoustic imaging: a review. *Journal of Biomedical Optics*, 17(6):061202–1–061202–22, 2012.

- [38] B. T. Cox, S. R. Arridge, and P. C. Beard. Photoacoustic tomography with a limited-aperture planar sensor and a reverberant cavity. *Inverse Problems*, 23(6):S95–S112, 2007.
- [39] B. T. Cox and P. C. Beard. Photoacoustic tomography with a single detector in a reverberant cavity. *The Journal of the Acoustical Society of America*, 125(3):1426–1436, 2009.
- [40] B. T. Cox, J. G. Laufer, and P. C. Beard. The challenges for quantitative photoacoustic imaging. *Proc. of SPIE*, 7177:717713, 2009.
- [41] A. Denisjuk. Integral geometry on the family of semi-spheres. *Fract. Calc. Appl. Anal.*, 2(1):31–46, 1999.
- [42] R. Ellwood, E. Zhang, P. Beard, and B. Cox. Photoacoustic imaging using acoustic reflectors to enhance planar arrays. *Journal of Biomedical Optics*, 19(12):126012, 2014.
- [43] J. A. Fawcett. Inversion of n -dimensional spherical averages. *SIAM J. Appl. Math.*, 45(2):336–341, 1985.
- [44] D. Finch, M. Haltmeier, and Rakesh. Inversion of spherical means and the wave equation in even dimensions. *SIAM J. Appl. Math.*, 68(2):392–412, 2007.
- [45] D. Finch, M. Haltmeier, and Rakesh. Inversion of spherical means and the wave equation in even dimensions. *SIAM J. Appl. Math.*, 2:392–412, 2007.
- [46] D. Finch, S. K. Patch, and Rakesh. Determining a function from its mean values over a family of spheres. *SIAM J. Math. Anal.*, 35(5):1213–1240 (electronic), 2004.
- [47] D. Finch and Rakesh. The spherical mean value operator with centers on a sphere. *Inverse Problems*, 23(6):S37–S49, 2007.
- [48] D. Finch and Rakesh. Recovering a function from its spherical mean values in two and three dimensions. in *Photoacoustic imaging and spectroscopy L. H. Wang (Editor)*, CRC Press, 2009.
- [49] B. Gebauer and O. Scherzer. Impedance-acoustic tomography. *SIAM J. Applied Math.*, 69(2):565–576, 2009.
- [50] J.-L. Gennisson, S. Catheline, S. Chaffai, and M. Fink. Transient elastography in anisotropic medium: Application to the measurement of slow and fast shear wave speeds in muscles. *J. Acoust. Soc. Am.*, 114:536–541, 2003.
- [51] M. Haltmeier. Inversion of circular means and the wave equation on convex planar domains. *Comput. Math. Appl.*, 65(7):1025–1036, 2013.
- [52] M. Haltmeier and S. Pereverzyev, Jr. Recovering a function from circular means or wave data on the boundary of parabolic domains. *SIAM J. Imaging Sci.*, 8(1):592–610, 2015.
- [53] M. Haltmeier and S. Pereverzyev, Jr. The universal back-projection formula for spherical means and the wave equation on certain quadric hypersurfaces. *J. Math. Anal. Appl.*, 429(1):366–382, 2015.
- [54] M. Haltmeier, O. Scherzer, P. Burgholzer, R. Nuster, and G. Paltauf. Thermoacoustic tomography and the circular Radon transform: exact inversion formula. *Math. Models Methods Appl. Sci.*, 17(4):635–655, 2007.
- [55] B. Holman and L. Kunyansky. Gradual time reversal in thermo- and photo-acoustic tomography within a resonant cavity. *Inverse Problems*, 31(3):035008, 2015.
- [56] A. Homan. Multi-wave imaging in attenuating media. *Inverse Probl. Imaging*, 7(4):1235–1250, 2013.
- [57] Y. Hristova. Time reversal in thermoacoustic tomography—an error estimate. *Inverse Problems*, 25(5):055008, 2009.

- [58] Y. Hristova, P. Kuchment, and L. Nguyen. Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media. *Inverse Problems*, 24(5):055006, 2008.
- [59] S. Kim, O. Kwon, J. K. Seo, and J.-R. Yoon. On a Nonlinear Partial Differential Equation Arising in Magnetic Resonance Electrical Impedance Tomography. *SIAM J. Math. Anal.*, 34(3):511–526, 2002.
- [60] R. A. Kruger, L. P., F. Y. R., and C. R. Appledorn. Photoacoustic ultrasound (PAUS) reconstruction tomography. *Med. Phys.*, 22:1605–1609, 1995.
- [61] R. A. Kruger, R. D. R., and K. G. A. Thermoacoustic computed tomography - technical considerations. *Med. Phys.*, 26:1832–1837, 1999.
- [62] K. Krupchyk, M. Lassas, and G. Uhlmann. Inverse problems with partial data for a magnetic Schrödinger operator in an infinite slab and on a bounded domain. *Comm. Math. Phys.*, 312(1):87–126, 2012.
- [63] P. Kuchment and L. Kunyansky. Mathematics of thermoacoustic tomography. *European J. Appl. Math.*, 19(2):191–224, 2008.
- [64] P. Kuchment and D. Steinhauer. Stabilizing inverse problems by internal data. *Inverse Problems*, 28(8):084007, 2012.
- [65] L. Kunyansky. Reconstruction of a function from its spherical (circular) means with the centers lying on the surface of certain polygons and polyhedra. *Inverse Problems*, 27(2):025012, 2011.
- [66] L. Kunyansky. Fast reconstruction algorithms for the thermoacoustic tomography in certain domains with cylindrical or spherical symmetries. *Inverse Probl. Imaging*, 6(1):111–131, 2012.
- [67] L. Kunyansky, B. Holman, and B. T. Cox. Photoacoustic tomography in a rectangular reflecting cavity. *Inverse Problems*, 29(12):125010, 2013.
- [68] L. A. Kunyansky. Explicit inversion formulae for the spherical mean Radon transform. *Inverse Problems*, 23(1):373–383, 2007.
- [69] L. A. Kunyansky. A series solution and a fast algorithm for the inversion of the spherical mean Radon transform. *Inverse Problems*, 23(6):S11–S20, 2007.
- [70] L. A. Kunyansky. Thermoacoustic tomography with detectors on an open curve: an efficient reconstruction algorithm. *Inverse Problems*, 24(5):055021, 2008.
- [71] X. Li and G. Uhlmann. Inverse problems with partial data in a slab. *Inverse Probl. Imaging*, 4(3):449–462, 2010.
- [72] K. Lin and J. R. McLaughlin. An error estimate on the direct inversion model in shear stiffness imaging. *Inverse Problems*, 25:075003, 2009.
- [73] V. Y. Lin and A. Pinkus. Approximation of multivariate functions. In *Advances in computational mathematics (New Delhi, 1993)*, volume 4 of *Ser. Approx. Decompos.*, pages 257–265. World Sci. Publ., River Edge, NJ, 1994.
- [74] J. McLaughlin, D. Renzi, J. Yoon, R. Ehman, and A. Manduca. Variance controlled shear stiffness images for mre data. In *Biomedical Imaging: Nano to Macro, 2006. 3rd IEEE International Symposium on*, pages 960–963, April 2006.
- [75] J. R. McLaughlin and J. Yoon. Unique identifiability of elastic parameters from time-dependent interior displacement measurement. *Inverse Problems*, 20:2545, 2004.
- [76] J. R. McLaughlin, N. Zhang, and A. Manduca. Calculating tissue shear modulus and pressure by 2D log-elastographic methods. *Inverse Problems*, 26(8):085007, 2010.

- [77] F. Monard and G. Bal. Inverse anisotropic diffusion from power density measurements in two dimensions. *Inverse Problems*, 28:084001, 2012.
- [78] F. Monard and G. Bal. Inverse diffusion problem with redundant internal information. *Inverse Problems and Imaging*, 6(2):289–313, 2012.
- [79] F. Monard and G. Bal. Inverse anisotropic conductivity from power density measurements in dimensions $n \geq 3$. *Comm. Partial Differential Equations*, 38(7):1183–1207, 2013.
- [80] R. Muthupillai, D. Lomas, P. Rossman, J. Greenleaf, A. Manduca, and R. Ehman. Magnetic resonance elastography by direct visualization of propagating acoustic strain waves. *Science*, 269(5232):1854–1857, 1995.
- [81] A. Nachman, A. Tamasan, and A. Timonov. Conductivity imaging with a single measurement of boundary and interior data. *Inverse Problems*, 23:2551–2563, 2007.
- [82] A. Nachman, A. Tamasan, and A. Timonov. Recovering the conductivity from a single measurement of interior data. *Inverse Problems*, 25:035014, 2009.
- [83] A. Nachman, A. Tamasan, and A. Timonov. Current density impedance imaging. *Tomography and Inverse Transport Theory. Contemporary Mathematics (G. Bal, D. Finch, P. Kuchment, P. Stefanov, G. Uhlmann, Editors)*, 559, 2011.
- [84] F. Natterer. Photo-acoustic inversion in convex domains. *Inverse Probl. Imaging*, 6(2):315–320, 2012.
- [85] L. V. Nguyen. A family of inversion formulas in thermoacoustic tomography. *Inverse Probl. Imaging*, 3(4):649–675, 2009.
- [86] S. J. Norton. Reconstruction of a two-dimensional reflecting medium over a circular domain: exact solution. *J. Acoust. Soc. Amer.*, 67(4):1266–1273, 1980.
- [87] S. J. Norton and M. Linzer. Ultrasonic reflectivity imaging in three dimensions: Exact inverse scattering solutions for plane, cylindrical, and spherical apertures. *Biomedical Engineering, IEEE Transactions on*, BME-28(2):202–220, Feb 1981.
- [88] J. Ophir, I. Céspedes, H. Ponnekanti, Y. Yazdi, and X. Li. Elastography: A quantitative method for imaging the elasticity of biological tissues. *Ultrasonic Imaging*, 13:111–134, 1991.
- [89] A. A. Oraevsky, S. L. Jacques, R. O. Esenaliev, and F. K. Tittel. Laser-based optoacoustic imaging in biological tissues. *Proc. SPIE*, 2134A:122–128, 1994.
- [90] V. Palamodov. *Reconstructive integral geometry*, volume 98 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2004.
- [91] V. P. Palamodov. A uniform reconstruction formula in integral geometry. *Inverse Problems*, 28(6):065014, 2012.
- [92] G. Paltauf, R. Nuster, M. Haltmeier, and P. Burgholzer. Experimental evaluation of reconstruction algorithms for limited view photoacoustic tomography with line detectors. *Inverse Problems*, 23(6):S81–S94, 2007.
- [93] S. Patch and O. Scherzer. Photo- and thermo- acoustic imaging. *Inverse Problems*, 23:S1–10, 2007.
- [94] D. A. Popov and D. V. Sushko. A parametrix for a problem of optical-acoustic tomography. *Dokl. Akad. Nauk*, 382(2):162–164, 2002.
- [95] D. A. Popov and D. V. Sushko. Image restoration in optical-acoustic tomography. *Problemy Peredachi Informatsii*, 40(3):81–107, 2004.
- [96] A. Pulkkinen, B. T. Cox, S. R. Arridge, J. P. Kaipio, and T. Tarvainen. A Bayesian approach to spectral quantitative photoacoustic tomography. *Inverse Problems*, 30(6):065012, 2014.

- [97] J. Qian, P. Stefanov, G. Uhlmann, and H. Zhao. An efficient Neumann series-based algorithm for thermoacoustic and photoacoustic tomography with variable sound speed. *SIAM J. Imaging Sci.*, 4(3):850–883, 2011.
- [98] Y. Salman. An inversion formula for the spherical mean transform with data on an ellipsoid in two and three dimensions. *J. Math. Anal. Appl.*, 420(1):612–620, 2014.
- [99] O. Scherzer, editor. *Handbook of Mathematical Methods in Imaging*. Springer Verlag, New York, 2011.
- [100] J. K. Seo and E. J. Woo. Magnetic Resonance Electrical Impedance Tomography (MREIT). *SIAM Review*, 53:40–68, 2011.
- [101] J. K. Seo and E. J. Woo. Magnetic resonance elastography. In *Nonlinear Inverse Problems in Imaging*, pages 335–350. John Wiley & Sons, 2012.
- [102] E. Somersalo, M. Cheney, and D. Isaacson. Existence and uniqueness for electrode models for electric current computed tomography. *SIAM J. Appl. Math.*, 52(4):1023–1040, 1992.
- [103] P. Stefanov and G. Uhlmann. Thermoacoustic tomography with variable sound speed. *Inverse Problems*, 25(7):075011, 2009.
- [104] P. Stefanov and G. Uhlmann. Thermoacoustic tomography arising in brain imaging. *Inverse Problems*, 27(4):045004, 2011.
- [105] P. Stefanov and G. Uhlmann. Multi-wave methods by ultrasounds. *Inside out, Cambridge University Press (G. Uhlmann, Ed.)*, 2012.
- [106] T. Tarvainen, A. Pulkkinen, B. Cox, J. Kaipio, and S. Arridge. Bayesian image reconstruction in quantitative photoacoustic tomography. *Medical Imaging, IEEE Transactions on*, 32(12):2287–2298, Dec 2013.
- [107] V. V. Tuchin, editor. *Handbook of optical biomedical diagnostics*, volume PM107. SPIE, Bellingham, WA, 2002.
- [108] G. Uhlmann. Calderón’s problem and electrical impedance tomography. *Inverse Problems*, 25:123011, 2009.
- [109] T. Vo-Dinh, editor. *Biomedical Photonics Handbook*. CRC press, 2003.
- [110] K. Wang and M. A. Anastasio. Photoacoustic and thermoacoustic tomography: Image formation principles. In O. Scherzer, editor, *Handbook of Mathematical Methods in Imaging*, pages 781–815. Springer New York, 2011.
- [111] L. V. Wang, editor. *Photoacoustic imaging and spectroscopy*. CRC Press, 2009.
- [112] L. V. Wang and H. Wu. *Biomedical Optics: Principles and Imaging*. Wiley, 2007.
- [113] M. Xu and L. V. Wang. Universal back-projection algorithm for photoacoustic computed tomography. *Phys. Rev. E*, 71:016706, Jan 2005.