A pointwise cubic average for two commuting transformations joint work with Wenbo Sun.

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1. Context

A fundamental question in ergodic theory is the convergence (in some sense) of the average

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n}x)\cdots f_d(T^{dn}x).$$

Here $(X,\mathcal{X},\mu)$ is a probability space and $T\colon X\to X$ is measurable, measure-preserving $(\mu(T^{-1}A)=\mu(A),\ A\in\mathcal{X})$.

Motivated by Furstenberg's proof (1977) of Szemerédi Theorem (which uses “characteristic” factors) .

Linked to additive combinatorics and number theory.

Some historical results:

L^2 convergence:

Furstenberg (1977): $d = 2$.

Conze and Lesigne (1988), Host and Kra (2001): $d = 3$.

Host and Kra (2005): $d \in \mathbb{N}$.

Several transformations

$(X, \mathcal{X}, \mu)$ probability space, $T_1 \ldots, T_d \colon X \to X$ measure preserving transformations.

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L^2 convergence:

Tao (2007, finitary), Towsner (2008, non-standard analysis), Austin (2009, ergodic), Host (2009, ergodic: using "cubes"): Commutative case.

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L^2 *convergence*:

Tao (2007, finitary), Towsner (2008, non-standard analysis), Austin (2009, ergodic), Host (2009, ergodic: using "cubes"): Commutative case.

Walsh (2012, finitary) : Nilpotent case.

Problems of pointwise and L^2 convergence are really different.

Pointwise convergence:

Pointwise convergence results

Problems of pointwise and L^2 convergence are really different.

Pointwise convergence:

Bourgain (1990) :

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x).$$

Huang, Shao and Ye (2014) : distal case.

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{n}x)f_2(T^{2n}x)\cdots f_d(T^{dn}x).$$

Proof using very special (in the sense of particular properties of some structures) topological models.

Cubic averages

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(T^i x)\cdot f_2(T^j x)\cdot f_3(T^{i+j} x).$$

$$\frac{1}{N^3}\sum_{i,j,k=0}^{N-1} f_1(T^i x) f_2(T^j x) f_3(T^{i+j} x) f_4(T^k x) f_5(T^{i+k} x) f_6(T^{j+k} x) f_7(T^{i+j+k} x).$$

L^2 convergence:

Bergelson (2000): $d=2$.
Host and Kra (2005): $d \in \mathbb{N}$.

Pointwise convergence:

Assani (2010), Chu and Frantzikinakis (2010), Huang, Shao and Ye (2014): $d \in \mathbb{N}$.

Cubic averages for several transformations

$(X, \mathcal{B}, \mu)$, $T_1, T_2, T_3 \colon X \to X$ measure preserving.

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$$\frac{1}{N^2} \sum_{i,j=0}^{N-1} f_1(T_1^i x) \cdot f_2(T_2^j x) \cdot f_3(T_3^{i+j} x). \tag{1}$$

$$\frac{1}{N^2} \sum_{i,j=0}^{N-1} f_1(T_1^i x) \cdot f_2(T_2^j x) \cdot f_3(T_1^i T_2^j x). \tag{2}$$

Cubic averages for several transformations

$(X,\mathcal{B},\mu)$, $T_1, T_2, T_3\colon X \to X$ measure preserving.

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(T_1^i x)\cdot f_2(T_2^j x)\cdot f_3(T_3^{i+j}x). \tag{1}$$

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(T_1^i x)\cdot f_2(T_2^j x)\cdot f_3(T_1^i T_2^j x). \tag{2}$$

Pointwise convergence Eq (1)

Assani (2007): $d = 3$.
Chu and Frantzikinakis (2010): $d \in \mathbb{N}$.
No commutativity required!!

Leibman (2002): Convergence of Eq (2) fails (even in L^2) with no commutativity assumptions.

Theorem (D., Sun (2014))

Let (X, μ, S, T) be a measure preserving system with commuting transformations S and T, ergodic for $\langle S, T\rangle$. Then for any $f_1, f_2, f_3 \in L^\infty(\mu)$ the average

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^i x) f_2(T^j x) f_3(S^i T^j x)$$

converges for a.e. $x \in X$ as N goes to infinity.

2. Proof ideas.

Recall (one single transformation)

A measure preserving system $(X, \mathcal{X}, \mu, T)$
probability space $(X, \mathcal{X}, \mu)$.
$T : X \to X$ measurable, measure-preserving
$(\mu(T^{-1}A) = \mu(A),\ A \in \mathcal{X})$.
$(X, \mathcal{X}, \mu, T)$ is ergodic if $A = T^{-1}A$ implies $\mu(A) = 0$ or 1.

A topological dynamical system (X, T)
X compact metric space.
$T : X \to X$ homeomorphism.
(X, T) is minimal if $\{T^n x : n \in \mathbb{Z}\}$ is dense for any $x \in X$.

Recall: factor map

Measurable

$$\begin{array}{ccc} (X,\mu) & \xrightarrow{\ T\ } & (X,\mu) \\ \pi \downarrow & & \downarrow \pi \\ (Y,\nu) & \xrightarrow[\ T\ ]{} & (Y,\nu) \end{array}$$

$\pi\mu = \nu$ and $\pi \circ T = T \circ \pi$.

Topological

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow[\ T\ ]{} & Y \end{array}$$

π onto, continuous and
$\pi \circ T = T \circ \pi$.

Strictly ergodic model

Theorem (Jewett, Krieger, 1969-1971)

Every ergodic (X, μ, T) has a strictly ergodic topological model.

$(\widehat{X}, T)$ is strictly ergodic if it is minimal and there exists a unique measure ergodic for T.

$(\widehat{X}, T)$ is a strictly ergodic topological model if $(\widehat{X}, T)$ is strictly ergodic with measure $\widehat{\mu}$ such that

$$(X, \mu, T) \cong (\widehat{X}, \widehat{\mu}, T).$$

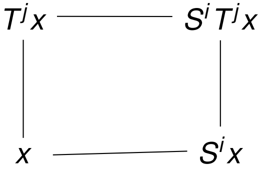
Jewett-Krieger theorem is also valid for (free)commutative group actions (Weiss (1985)) and more general, for (free)amenable group actions (Weiss-Rosenthal, never published!)

Recall (from Wenbo's talk)

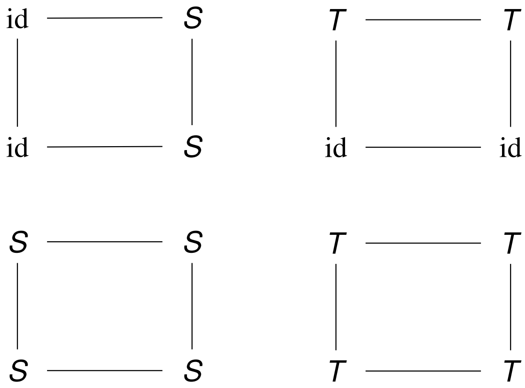
Let (X, S, T) be a system with commuting transformations.$\mathbf{Q}_{S,T}$ is the set

$$\overline{\{(x, S^i x, T^i x, S^i T^i x) : x \in X, i, j \in \mathbb{Z}\}}.$$

An element in $\mathbf{Q}_{S,T}$ is called a dynamical cube.



$\mathbf{Q}_{S,T}$ is invariant under the transformations:



Passing to a topological model problem

$(\widehat{X}, S, T)$ is a $\mathbf{Q}_{S,T}$-strictly ergodic model for (X, μ, S, T) if $(\widehat{X}, S, T)$ is a strictly ergodic model and $\mathbf{Q}_{S,T}(\widehat{X})$ is strictly ergodic.

Passing to a topological model problem

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Proposition

If (X, μ, S, T) has a $\mathbf{Q}_{S,T}$-strictly ergodic model, then the average

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=0}^{N-1} f_1(S^i x) \cdot f_2(T^j x) \cdot f_3(S^i T^j x)$$

converges for μ-a.e. x.

Question

Does (X, μ, S, T) have a $\mathbf{Q}_{S,T}(X)$-strictly ergodic model?

Finding a $\mathbf{Q}_{S,T}$-strictly ergodic model

$$\mathbf{Q}_{S,T}(\widehat{X}) = \overline{\{(x, S^i x, T^j x, S^i T^j x) : x \in \widehat{X}, i, j \in \mathbb{Z}\}}.$$

Question

Does (X, μ, S, T) have a $\mathbf{Q}_{S,T}$-strictly ergodic model?

$S = T$: Huang, Shao, Ye (2014).
$S \neq T$: ???

Theorem (D., Sun)

There exists an extension (X', μ', S, T) of (X, μ, S, T) which has a $\mathbf{Q}_{S,T}$-strictly ergodic model.

Lift the property from a factor

Proposition

There exists an extension X' of X such that $\mathcal{Z}_{S,T}(X')$ has a $\mathbf{Q}_{S,T}$-strictly ergodic model.

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There exists an extension X' of X such that $\mathcal{Z}_{S,T}(X')$ has a $\mathbf{Q}_{S,T}$-strictly ergodic model.

The idea is to lift the property from a "characteristic" factor $\mathcal{Z}_{S,T}$ (which comes from the L^2 convergence).

Theorem (Host(2009))

There exists a factor $\mathcal{Z}_{S,T}$ of X' such that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=0}^{N-1} f_1(S^i x) \cdot f_2(T^j x) \cdot f_3(S^i T^j x)$$

$$= \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=0}^{N-1} f_1(S^i x) \cdot f_2(T^j x) \cdot S^i T^j \mathbb{E}(f_3 | \mathcal{Z}_{S,T})(x)$$

in the $L^2(\mu)$ sense.

- *magic*: $\mathcal{Z}_{S,T}(X') = \mathcal{I}_S(X') \vee \mathcal{I}_T(X')$ (Host (2009)).
- $\mathcal{I}_S(X') \vee \mathcal{I}_T(X')$ has a $\mathbf{Q}_{S,T}$-strictly ergodic model.

Proposition

X' can be taken ergodic and free.

For example (X, μ, T, T^2) has a magic,ergodic and free extension.

$\mathbf{Q}_{S,T}$-strictly ergodic model for $\mathcal{I}_S \vee \mathcal{I}_T$

Proposition

$$(\mathcal{I}_S(X') \vee \mathcal{I}_T(X'), \mu, S, T) \cong (\mathcal{I}_T(X') \times \mathcal{I}_S(X'), \mu \times \mu, S \times \mathrm{id}, \mathrm{id} \times T).$$

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- $\mathcal{I}_S(X') \vee \mathcal{I}_T(X') \leftrightarrow \mathcal{I}_T(X') \times \mathcal{I}_S(X')$
- $A \cap B \leftrightarrow B \times A, \forall SA = A, TB = B.$
- $\mu(A \cap B) = \mu(A)\mu(B), \ \forall SA = A, TB = B.$
  $\mu(A \cap B) = \frac{1}{N^2} \sum 1_A(S^i T^j x) 1_B(S^i T^j x) = \frac{1}{N} \sum 1_A(T^j x) \frac{1}{N} \sum 1_B(S^i x) = \mu(A)\mu(B).$
- $S \leftrightarrow S \times \mathrm{id}$
- $T \leftrightarrow \mathrm{id} \times T$

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Proposition

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- $\mathcal{I}_S(X') \vee \mathcal{I}_T(X') \leftrightarrow \mathcal{I}_T(X') \times \mathcal{I}_S(X')$
- $A \cap B \leftrightarrow B \times A, \forall SA = A, TB = B.$
- $\mu(A \cap B) = \mu(A)\mu(B), \ \ \forall SA = A, TB = B.$
  $\mu(A \cap B) = \frac{1}{N^2} \sum 1_A(S^i T^j x) 1_B(S^i T^j x) = \frac{1}{N} \sum 1_A(T^j x) \frac{1}{N} \sum 1_B(S^i x) = \mu(A)\mu(B).$
- $S \leftrightarrow S \times \mathrm{id}$
- $T \leftrightarrow \mathrm{id} \times T$

Proposition

Every product system $(Y \times W, \mu \times \nu, S \times \mathrm{id}, \mathrm{id} \times T)$ *has a* $\mathbf{Q}_{S,T}$*-strictly ergodic model.*

$$\mathbf{Q}_{S,T}(Y \times W) =$$

$$\overline{\{((y,w),(S^i y,w),(y,T^j w),(S^i y,T^j w)) : (y,w) \in Y \times W, i,j \in \mathbb{Z}\}}$$

$$\begin{array}{ccc} (y,w') & \text{———} & (y',w') \\ | & & | \\ (y,w) & \text{———} & (y',w) \end{array}$$

$y, y' \in Y, w, w' \in W$.

$\mathbf{Q}_{S,T}(Y \times W)$ is invariant under the transformations:

$$\begin{array}{ccc} \mathrm{id}\times\mathrm{id} & \text{———} & S\times\mathrm{id} \\ | & & | \\ \mathrm{id}\times\mathrm{id} & \text{———} & S\times\mathrm{id} \end{array} \qquad \begin{array}{ccc} \mathrm{id}\times T & \text{———} & \mathrm{id}\times T \\ | & & | \\ \mathrm{id}\times\mathrm{id} & \text{———} & \mathrm{id}\times\mathrm{id} \end{array}$$

$$\begin{array}{ccc} S\times\mathrm{id} & \text{———} & S\times\mathrm{id} \\ | & & | \\ S\times\mathrm{id} & \text{———} & S\times\mathrm{id} \end{array} \qquad \begin{array}{ccc} \mathrm{id}\times T & \text{———} & \mathrm{id}\times T \\ | & & | \\ \mathrm{id}\times T & \text{———} & \mathrm{id}\times T \end{array}$$

$\mathbf{Q}_{S,T}(Y \times W)$ is invariant under the transformations:

$$\begin{array}{ccc} \mathrm{id}\times\mathrm{id} & \text{———} & S\times\mathrm{id} \\ | & & | \\ \mathrm{id}\times\mathrm{id} & \text{———} & S\times\mathrm{id} \end{array} \qquad \begin{array}{ccc} \mathrm{id}\times T & \text{———} & \mathrm{id}\times T \\ | & & | \\ \mathrm{id}\times\mathrm{id} & \text{———} & \mathrm{id}\times\mathrm{id} \end{array}$$

$$\begin{array}{ccc} S\times\mathrm{id} & \text{———} & \mathrm{id}\times\mathrm{id} \\ | & & | \\ S\times\mathrm{id} & \text{———} & \mathrm{id}\times\mathrm{id} \end{array} \qquad \begin{array}{ccc} \mathrm{id}\times\mathrm{id} & \text{———} & \mathrm{id}\times\mathrm{id} \\ | & & | \\ \mathrm{id}\times T & \text{———} & \mathrm{id}\times T \end{array}$$

Identify

$$\begin{array}{ccc} (y, w') & \text{———} & (y', w') \\ | & & | \\ (y, w) & \text{———} & (y', w) \end{array}$$

with (y, y', w, w').

$\mathbf{Q}_{S,T}(Y \times W) = Y \times Y \times W \times W$ is invariant under $S \times \mathrm{id} \times \mathrm{id} \times \mathrm{id}$, $\quad \mathrm{id} \times S \times \mathrm{id} \times \mathrm{id}$, $\quad \mathrm{id} \times \mathrm{id} \times T \times \mathrm{id}$ $\quad$ and $\mathrm{id} \times \mathrm{id} \times \mathrm{id} \times T$.

Final ingredient: The relative Jewett-Krieger Theorem

Theorem (Weiss (1985))

Let $\pi\colon X \to Z$ a (measurable) factor map. For any strictly ergodic model $\widehat{Z}$ of Z there exists a strictly ergodic model $\widehat{X}$ of X and a topological factor map $\widehat{\pi}\colon \widehat{X} \to \widehat{Z}$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ \Phi\ } & \widehat{X} \\ \pi \downarrow & & \downarrow \widehat{\pi} \\ Z & \xrightarrow[\ \phi\ ]{} & \widehat{Z} \end{array}$$

the diagram commutes.

$$
\begin{array}{ccccc}
X' & \xrightarrow{\Phi} & \widehat{X} & \xleftarrow{p_0} & \mathbf{Q}_{S,T}(\widehat{X}) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \widehat{\pi}^4} \\
\mathcal{Z}_{S,T} & \xrightarrow[\phi]{} & Y \times W & \xleftarrow[p_0]{} & \mathbf{Q}_{S,T}(Y \times W)
\end{array}
$$

$$
\begin{array}{ccccc}
X' & \xrightarrow{\ \Phi\ } & \widehat{X} & \xleftarrow{\ p_0\ } & \mathbf{Q}_{S,T}(\widehat{X}) \\
\downarrow \pi & & \downarrow \widehat{\pi} & & \downarrow \widehat{\pi}^4 \\
\mathcal{Z}_{S,T} & \xrightarrow[\ \phi\ ]{} & Y \times W & \xleftarrow[\ p_0\ ]{} & \mathbf{Q}_{S,T}(Y \times W)
\end{array}
$$

$\mathbf{Q}_{S,T}(\widehat{X})$ is strictly ergodic.

3. Further questions

Higher order (topological) cubical spaces can help to deduce other cubic averages. For example $\mathbf{Q}_{T_1,T_2,T_3}(X)$ should help to prove the pointwise convergence of

$$\frac{1}{N^3}\sum_{\substack{0\leq n<N\\ 0\leq m<N\\ 0\leq p<N}} f_1(T_1^n x)f_2(T_2^m x)f_3(T_1^n T_2^m x)f_4(T_3^p x)f_5(T_1^n T_3^p x)f_6(T_2^m T_3^p x)f_7(T_1^n T_2^m T_3^p x)$$

for bounded functions $f_1, f_2, f_3, f_4, f_5, f_6, f_7$.

A related question is the pointwise convergence of

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(S^n x) f_2(T^n x).$$

One can study the strict ergodicity of the structure

$$N_{S,T}(X) = \overline{\{(x, S^n x, T^n x) : x \in X, n \in \mathbb{Z}\}}$$

under the transformations $\mathrm{id} \times S \times T$, $S \times S \times S$ and $T \times T \times T$ to obtain partial results.

Thank you for your attention!