A general van der Corput lemma and underlying Ramsey theory

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Agenda of the talk

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2. **Main result:** a van der Corput lemma for a general class of filters on semigroups, called ∂-filters
3. **Behind the scene:** a Ramsey theorem for ∂-filters

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- **Limit along $\mathcal{F}$:** for a topological space X, a sequence $(x_g)_{g \in G}$ and $x \in X$, we write $\lim_{g \to \mathcal{F}} x_g = x$ if for every open neighborhood $U \ni x$, we have $\forall^{\mathcal{F}} g \in G (x_g \in U)$.

An example to keep in mind

For $G = \mathbb{N}$ (or any amenable group), define **lower density** for subsets $A \subseteq \mathbb{N}$ by

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The sets of (lower) density 1 form the so-called **density filter** $\mathcal{F}_d$.

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- Let $\mathcal{F}$ be a filter on G.
- The action is called **mixing along** $\mathcal{F}$ if for every $f_0, f_1 \in L^2(X,\nu)$,

$$\lim_{g\to\mathcal{F}} \langle f_0, g \cdot_\alpha f_1 \rangle = \lim_{g\to\mathcal{F}} \int_X f_0(g \cdot_\alpha f_1) = \int_X f_0 \int_X f_1.$$

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- Equivalently, for sets $A, B \subseteq X$, take $f_0 = \chi_A, f_1 = \chi_B$, so

$$\lim_{g \to \mathcal{F}} \nu(A \cap B \cdot_\alpha g^{-1}) = \nu(A)\nu(B)$$

i.e. A and $B \cdot_\alpha g^{-1}$ gradually become probabilistically independent.

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In particular, if $A = B = C$ and A is ν-positive, then

$$\forall^{\mathcal{F}} g \in G \quad \nu\big(A \cap [g^{-1} \cdot_\alpha A] \cap [g^{-1} \cdot_\beta A]\big) > 0.$$

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So we have a sequence $(e_g)_{g \in G}$ in a Hilbert space $\mathcal{H} = L^2(X, \nu)$ and we need to understand when we'd have $\lim_{g \to \mathcal{F}} \langle f, e_g \rangle = 0$, for every $f \in \mathcal{H}$.

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We formulate the general version of this phenomenon as a property of a triple $(G, \mathcal{P}, \mathcal{F})$, where $\mathcal{F}$ is a filter on the semigroup and $\mathcal{P}$ is some collection of subsets of G, which are presumably "measurable", i.e. nice.

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The van der Corput property

The triple $(G, \mathcal{P}, \mathcal{F})$ is said to have the van der Corput property if for any weakly $\mathcal{P}$-measurable bounded sequence $(e_g)_{g \in G}$ in a Hilbert space $\mathcal{H}$, we have

$$\lim_{h \to \mathcal{F}} \lim_{g \to \mathcal{F}} \langle e_g, e_{gh} \rangle = 0 \implies \lim_{g \to \mathcal{F}} \langle f, e_g \rangle = 0, \ \forall f \in \mathcal{H}.$$

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- Recall that we were left with Case 2: $\int_X f_2 = 0$, and double recurrence boiled down to

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\text{[writing neatly]} &= \langle g\cdot_\alpha F_1^{(h)}, g\cdot_\beta F_2^{(h)}\rangle
\end{aligned}$$

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\langle e_g, e_{gh}\rangle &= \textstyle\int_X (g\cdot_\alpha f_1)(g\cdot_\beta f_2)(g\cdot_\alpha h\cdot_\alpha f_1)(g\cdot_\beta h\cdot_\beta f_2)\\
\text{[regrouping]} &= \textstyle\int_X (g\cdot_\alpha [f_1(h\cdot_\alpha f_1)])(g\cdot_\beta [f_2(h\cdot_\beta f_2)])\\
\text{[writing neatly]} &= \langle g\cdot_\alpha F_1^{(h)}, g\cdot_\beta F_2^{(h)}\rangle\\
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\end{aligned}$$

Single recurrence $\xrightarrow{\text{van der Corput}}$ double recurrence

- Recall that we were left with Case 2: $\int_X f_2 = 0$, and double recurrence boiled down to

$$\lim_{g\to\mathcal{F}} \langle f_0, e_g\rangle = \lim_{g\to\mathcal{F}} \int_X f_0 (g \cdot_\alpha f_1)(g \cdot_\beta f_2) = 0$$

- Suppose that $(G, \mathcal{P}, \mathcal{F})$ has the van der Corput property

$$\lim_{h\to\mathcal{F}} \lim_{g\to\mathcal{F}} \langle e_g, e_{gh}\rangle = 0 \implies \lim_{g\to\mathcal{F}} \langle f, e_g\rangle = 0, \text{ for all } f \in \mathcal{H},$$

where $\mathcal{P}$ is some rich enough collection of subsets of G so that $(e_g)_{g\in G}$ is weakly $\mathcal{P}$-measurable.

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\end{aligned}$$

Single recurrence $\xrightarrow{\text{van der Corput}}$ multiple recurrence

By induction, we get that single recurrence (i.e. mixing) implies multiple:

$$\lim_{g \to \mathcal{F}} \int_X f_0 (g \cdot_{\alpha_1} f_1)(g \cdot_{\alpha_2} f_2) \ldots (g \cdot_{\alpha_k} f_k) = \int_X f_0 \int_X f_1 \int_X f_2 \ldots \int_X f_k.$$

This is one of the key ingredients in Furstenberg's proof of Szemerédi's theorem.

Ramsey behind van der Corput

The van der Corput property for $(G, \mathcal{P}, \mathcal{F})$

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Say that $(G, \mathcal{L}, \mathcal{F})$ has the difference-Ramsey property if any graph $E \subseteq G^2$ satisfying $\forall^{\mathcal{F}} h \forall^{\mathcal{F}} g \ E(g, gh)$ contains arbitrarily large complete subgraphs in any set $A \in \mathcal{L}$,

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So far, all of the filters were **almost invariant**, i.e. for any $A \subseteq G$,

$$A \text{ is } \mathcal{F}\text{-large} \implies \forall^{\mathcal{F}} g \ (Ag^{-1} \text{ is } \mathcal{F}\text{-large}).$$

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An example of the van der Corput property with a non-almost-invariant filter

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So what is common between this and the previous examples?

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For any ∂-filter $\mathcal{F}$ on a semigroup G, the filtered L*-semigroup $(G, C^\infty(\mathcal{F}), \mathcal{F})$ has the difference-Ramsey property.*

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This immediately implies:

van der Corput lemma for ∂-filters (Ʊ.)

For any ∂-filter $\mathcal{F}$ on a semigroup G, the triple $(G, \check{\mathcal{F}} \cup C^\infty(\mathcal{F}), \mathcal{F})$ has the van der Corput property.

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Thus, our van der Corput lemma for ∂-filters indeed generalizes all of the previously mentioned instances.

THANK YOU