High piecewise syndeticity of product sets in amenable groups

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Combinatorics meets ergodic theory
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Jin's Theorem

Theorem (Jin, 2002)

If $A, B \subseteq \mathbb{Z}$ are such that $\mathrm{BD}(A), \mathrm{BD}(B) > 0$, then $A + B$ is piecewise syndetic: there is $m \in \mathbb{N}$ such that $A + B + [-m, m]$ contains intervals of arbitrarily large length.

The proof used *nonstandard analysis* and is, after looking through an appropriate "lens," almost identical to the proof that the sum of two sets of positive Lebesgue measure contains an interval.

A quantitative version of Jin's Theorem

Theorem (DGJLLM, 2013)

Suppose that $A, B \subseteq \mathbb{Z}$ are such that $\bar{d}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$. Then $A + B$ is upper syndetic of level α: there is $m \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, we have

$$\bar{d}(\{x \in \mathbb{Z} \ : \ x + [-k, k] \subseteq A + B + [-m, m]\}) \geq \alpha.$$

This proof also used nonstandard analysis. In particular, we had to formulate and prove a version of the Lebesgue Density Theorem for quotients of Loeb measure spaces. The proof actually works for subsets of $\mathbb{Z}^d$ for any d.

Amenable groups

Let G be a countable (discrete) group.

Definition

G is *amenable* if G admits a (left) Folner sequence, namely a sequence $\mathcal{S} := (S_n)$ of finite subsets of G such that, for every $g \in G$, we have

$$\lim_{n\to\infty} \frac{|gS_n \triangle S_n|}{|S_n|} = 0.$$

Examples of (countable) amenable groups include finite groups and virtually solvable groups. Free groups are not amenable.

Densities in amenable groups

Definition

Suppose that G is an amenable group and A is a subset of G.

1. If $\mathcal{S} = (S_n)$ is a Folner sequence for G, the (upper) $\mathcal{S}$-density of A is
$$d_{\mathcal{S}}(A) := \limsup_{n \to \infty} \frac{|A \cap S_n|}{|S_n|}.$$
2. The *Banach density* of A is
$$\mathrm{BD}(A) := \sup\{d_{\mathcal{S}}(A) \ : \ \mathcal{S} \text{ a Folner sequence for } G\}.$$

Remark

Suppose $G = \mathbb{Z}$. If $S_n = [-n, n]$, then $d_{\mathcal{S}} = \bar{d}$. One can also check that the notion of Banach density is the same.

Amenable group version of Jin's Theorem

Theorem (Beiglböck, Bergelson, Fish, 2009)

Suppose that G is a countable amenable group and $A, B \subseteq G$ are such that $\mathrm{BD}(A), \mathrm{BD}(B) > 0$. Then AB is piecewise syndetic: there is a finite set $E \subseteq G$ so that, for all finite sets $L \subseteq G$, there is $x \in G$ with $Lx \subseteq EAB$.

- This theorem was originally proven using ergodic theory.
- Later, Di Nasso and Lupini gave a combinatorial proof using nonstandard analysis that also worked for uncountable amenable groups and which showed one could assume $|E| \leq \lfloor \frac{1}{\mathrm{BD}(A)\,\mathrm{BD}(B)} \rfloor$.
- The point of this talk is to show how we can achieve a quantitative version of the above theorem generalizing our theorem for $\mathbb{Z}^d$.

$\mathcal{S}$-thick and $\mathcal{S}$-syndetic

Definition

Suppose that G is a countable amenable group, $\mathcal{S}$ is a Folner sequence for G, A is a subset of G, and $\alpha > 0$.

1. We say that A is $\mathcal{S}$-thick of level α if, for any finite $L \subseteq G$, we have
$$d_{\mathcal{S}}(\{x \in G \ : \ Lx \subseteq A\}) \geq \alpha.$$
2. We say that A is $\mathcal{S}$-syndetic of level α if there is a finite $E \subseteq G$ such that EA is $\mathcal{S}$-thick of level α.

The main results

Theorem (DGJLLM, 2015)

Suppose that G is a countable amenable group and $\mathcal{S}$ a Folner sequence. Further suppose that $A, B \subseteq G$ are such that $d_{\mathcal{S}}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$. Then BA is $\mathcal{S}$-syndetic of level α' for every $\alpha' < \alpha$. If, in addition, G is abelian, then BA is $\mathcal{S}$-syndetic of level α.

- When $G = \mathbb{Z}^d$ and $S_n = [-n, n]^d$, this recovers our earlier theorem.
- This also recovers another theorem of ours: if $A, B \subseteq \mathbb{Z}^d$ are such that $\underline{d}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$, then $A + B$ is $\mathcal{S}'$-syndetic for any subsequence $\mathcal{S}'$ of $\mathcal{S}$.
- The moreover part: we may assume that $d_{\mathcal{S}}(A \cup gA) > \alpha$ for some $g \in G$ (otherwise you can show that A is already $\mathcal{S}$-thick of level α) and that $\mathrm{BD}(B) > 1/2$, so $\mathrm{BD}(B \cap Bg^{-1}) > 0$.

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A few words about the setup

- Every set X is "logically completed" to its *nonstandard extension* X^*. (One can take X^* to be a nonprincipal ultrapower of X.)
- The logical completions contain "ideal" elements such as infinite elements and infinitesimal elements.
- X^* logically behaves like X (*transfer principle* or *Łos' theorem*) with respect to certain subsets of X^*, called the *internal subsets*. For example, nonempty internal subsets of $\mathbb{R}^*$ that are bounded above have suprema. (Thus, the set of infinitesimal elements of $\mathbb{R}^*$ is not internal.)
- Every element x of $\mathbb{R}^*$ that belongs to the convex hull of $\mathbb{Z}$ is within an infinitesimal distance of a unique element of $\mathbb{R}$, called the *standard part of* x, denoted $\operatorname{st}(x)$.
- If (r_n) is a sequence from $\mathbb{R}$ and $L \in \mathbb{R}$, then L is a subsequential limit of (r_n) if and only if there is $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ such that $\operatorname{st}(r_\nu) = L$.

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Hyperfinite sets

- A (necessarily internal) subset E of X^* is called *hyperfinite* if there is an internal bijection $f : E \to [1, \nu]$ for some $\nu \in \mathbb{N}^*$. This ν is unique and is called the *internal cardinality* of E, denoted $|E|$. (In the ultrapower setup, hyperfinite=ultraproduct of finite sets.)

Key Example

If (S_n) is a Folner sequence for G, then for any $\nu \in \mathbb{N}^*$, S_ν is a hyperfinite set. For any $A \subseteq G$, we have

$$d_S(A) := \max\left\{\operatorname{st}\left(\frac{|A^* \cap S_\nu|}{|S_\nu|}\right) \ : \ \nu > \mathbb{N}\right\}.$$

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Loeb measure

Suppose that E is a hyperfinite set. We can define a finitely additive measure μ_E on the collection of internal subsets of E given by

$$\mu_E(A) := \mathrm{st}\left(\frac{|A|}{|E|}\right).$$

μ_E then extends to a probability measure on the σ-algebra of *Loeb measurable* subsets of E.

For internal $C \subseteq G^*$, we write $\mu_E(C)$ instead of $\mu_E(C \cap E)$.

Folner approximations

Fact (Di Nasso and Lupini)

G is amenable if and only if G has a *Folner approximation*, which is a hyperfinite subset $Y \subseteq G^*$ such that, for all $g \in G$, we have

$$\frac{|gY \triangle Y|}{|Y|} \approx 0,$$

or, equivalently, for all $g \in G$, we have $\mu_Y(gY) = 1$. In this case, for any $A \subseteq G$, we have

$$\mathrm{BD}(A) = \max\{\mu_Y(A^*) \ : \ Y \text{ a Folner approximation for } G\}.$$

For example, if $\mathcal{S} = (S_n)$ is a Folner sequence for G, then S_ν is a Folner approximation for G whenever $\nu > \mathbb{N}$.

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The "process"

- Fix $\nu > \mathbb{N}$ and hyperfinite $\Gamma \subseteq S_\nu$ with $\mu_{S_\nu}(\Gamma) \geq \alpha$.
- We define a sequence (H_n) of subsets of G and a sequence (s_n) from G as follows.
- $H_0 := \{g \in G \ : \ \mu_\Gamma(\{x \in \Gamma \ : \ gx \in (BA)^*\}) < 1\}$.
- Suppose that H_n has been defined and is not empty. Let $s_n \in H_n$ be arbitrary and define

$$H_{n+1} := \{g \in G \ : \ \mu_\Gamma(\{x \in \Gamma \ : \ gx \in (\{s_0, \ldots, s_n\}BA)^*\}) < 1\}.$$

- Suppose there is n with $H_n = \emptyset$ and let $E := \{s_0, \ldots, s_{n-1}\}$. We claim that EBA is $\mathcal{S}$-thick of level α.
- Fix $L \subseteq G$ finite. Since $H_n = \emptyset$, we have that $Lx \subseteq (EBA)^*$ for almost all $x \in \Gamma$, whence

$$d_S(\{x \in G \ : \ Lx \subseteq EBA\}) \geq \mu_{S_\nu}(\{x \in S_\nu \ : \ Lx \subseteq (EBA)^*)\} \geq \alpha.$$

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The “process” (continued)

- We are seeking $\nu > \mathbb{N}$ and hyperfinite $\Gamma \subseteq S_\nu$ with $\mu_{S_\nu}(\Gamma) \geq \alpha$ such that the process stops at some finite stage.
- Suppose there is standard $r > 0$, $\nu > \mathbb{N}$, and Folner approximation Y of G such that, setting $C := A^* \cap S_\nu$, $D := B^* \cap Y$, and

$$\Gamma := \{x \in S_\nu \ : \ \frac{|xC^{-1} \cap D|}{|Y|} \geq r\},$$

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- We then claim that $H_n = \emptyset$ for $n > \frac{1}{r}$.

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- Suppose that $H_n \neq \emptyset$. For $k = 0, \ldots, n$, fix $s_k \in H_k$ and take $\gamma_k \in \Gamma$ so that $s_k\gamma_k \notin (\{s_0, \ldots, s_{k-1}\}BA)^*$.
- Note then that if $0 \leq i < j \leq n-1$, then $s_j\gamma_j(A^*)^{-1} \cap s_iB^* = \emptyset$.
- It follows that the sets $s_k((\gamma_k C^{-1}) \cap D)$ for $k = 0, \ldots, n-1$ are pairwise disjoint.
- Since Y is a Folner approximation for G, we have

$$1 \geq \mathrm{st}\left(\frac{|\bigcup_{k=0}^{n-1} s_k((\gamma_k C^{-1}) \cap D)|}{|Y|}\right) = \sum_{k=0}^{n-1} \mu_Y((\gamma_k C^{-1}) \cap D) \geq nr.$$

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- Note then that if $0 \leq i < j \leq n-1$, then $s_j\gamma_j(A^*)^{-1} \cap s_iB^* = \emptyset$.
- It follows that the sets $s_k((\gamma_k C^{-1}) \cap D)$ for $k = 0, \ldots, n-1$ are pairwise disjoint.
- Since Y is a Folner approximation for G, we have

$$1 \geq \operatorname{st}\left(\frac{|\bigcup_{k=0}^{n-1} s_k((\gamma_k C^{-1}) \cap D)|}{|Y|}\right) = \sum_{k=0}^{n-1} \mu_Y((\gamma_k C^{-1}) \cap D) \geq nr.$$

- It follows that $n \leq \frac{1}{r}$.

The key lemma

Lemma

Suppose $d_S(A) \geq \alpha$ *and* $\mathrm{BD}(B) \geq \beta$. *Then there exists a Folner approximation* Y *of* G *and* $\nu > \mathbb{N}$ *such that, setting* $C := A^* \cap S_\nu$ *and* $D := B^* \cap Y$, *we have:*

1. $\mu_{S_\nu}(C) \geq \alpha$;
2. $\mu_Y(D) \geq \beta$;
3. $\mathrm{st}\left(\frac{1}{|S_\nu|}\sum_{x\in S_\nu}\frac{|(xC^{-1})\cap D|}{|Y|}\right) \geq \alpha\beta$.

Finishing the process from the key lemma

- Recall $d_{\mathcal{S}}(A) > \alpha$ and $\mathrm{BD}(B) > 0$.
- Fact: there is a finite T such that $d_{\mathcal{S}}(A) \cdot \mathrm{BD}(TB) > \alpha$. Since BA is $\mathcal{S}$-syndetic of level α if and only if TBA is $\mathcal{S}$-syndetic of level α, we may suppose that $T = \{1\}$.
- Take Y and ν as in the previous lemma.
- Take standard $r > 0$ so that $d_{\mathcal{S}}(A) \cdot \mathrm{BD}(B) > \alpha + r$ and recall $\Gamma := \{x \in S_\nu \ : \ \frac{|(xC^{-1}) \cap D|}{|Y|} \geq r\}$.
- We then have

$$\alpha + r < \frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{|(xC^{-1}) \cap D|}{|Y|} \leq \frac{1}{|S_\nu|} \left(\sum_{x \in \Gamma} + \sum_{x \notin \Gamma} \right) \leq \frac{|\Gamma|}{|S_\nu|} + r.$$

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Proving the key lemma

- Take any Folner approximation Y for G such that $\mu_Y(D) \geq \beta$.
- Since $d_S(A) \geq \alpha$, the following statement is true: for any finite $E \subseteq G$ and any $n_0 \in \mathbb{N}$, there is $n > n_0$ such that $\frac{|A \cap S_n|}{|S_n|} \geq \alpha - 2^{-n_0}$ and for which, for all $g \in E$, we have $\frac{|g^{-1}S_n \triangle S_n|}{|S_n|} < 2^{-n_0}$.
- Apply the transferred version of this statement to Y and some given $\nu_0 > \mathbb{N}$ to get $\nu > \nu_0$ such that $\mu_{S_\nu}(C) \geq \alpha$ and for which $\frac{|g^{-1}S_\nu \triangle S_\nu|}{|S_\nu|} \approx 0$ for all $g \in E$.
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We finish by counting

$$
\begin{aligned}
\frac{1}{|S_\nu|}\sum_{x\in S_\nu}\frac{\left|xC^{-1}\cap D\right|}{|Y|} &= \frac{1}{|S_\nu|}\sum_{x\in S_\nu}\frac{1}{|Y|}\sum_{d\in D}\chi_{C^{-1}}(x^{-1}d)\\
&= \frac{1}{|Y|}\sum_{d\in D}\frac{1}{|S_\nu|}\sum_{x\in S_\nu}\chi_{C^{-1}}(x^{-1}d)\\
&= \frac{1}{|Y|}\sum_{d\in D}\frac{\left|S_\nu^{-1}d\cap C^{-1}\right|}{|S_\nu|}\\
&\geq \frac{1}{|Y|}\sum_{d\in D}\left(\frac{\left|C^{-1}\right|}{|S_\nu|}-\frac{\left|d^{-1}S_\nu\triangle S_\nu\right|}{|S_\nu|}\right)\\
&\geq \frac{|C|}{|S_\nu|}\frac{|D|}{|Y|}-\text{ an infinitesimal.}
\end{aligned}
$$

More detail on the moreover part

Lemma

If $d_{\mathcal{S}}(A \cup gA) = \alpha$ for all $g \in G$, then A is $\mathcal{S}$-thick of level α.

Proof.

- Take $\nu > \mathbb{N}$ so that $\mu_{S_\nu}(A^*) = \alpha$. Fix $E \subseteq G$ finite.
- Set $A_0 := \{x \in A^* \cap S_\nu \ : \ Ex \subseteq A^*\}$. We want that $\mu_{S_\nu}(A_0) = \alpha$.
- Set $R := (A^* \cap S_\nu) \setminus A_0$ and suppose that $\mu_{S_\nu}(R) > 0$.
- Let $x \mapsto g_x : R \to E$ be an internal mapping such that $g_x x \notin A^*$. There is $g \in E$ such that, setting $R_0 := \{x \in R \ : \ g_x = g\}$, we have $\mu_{S_\nu}(R_0) > 0$.
- It remains to observe that gR_0 is disjoint from A^*, whence:

$$d_{\mathcal{S}}(A \cup gA) \geq \mu_{S_\nu}(A^* \cup gR_0) = \mu_{S_\nu}(A^*) + \mu_{S_\nu}(gR_0) > \alpha.$$

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Lower density versions?

Theorem (DGJLLM, 2013)

Suppose that $A, B \subseteq \mathbb{Z}^d$.

1. *Suppose that* $\underline{d}(A) = \alpha > 0$ *and* $\mathrm{BD}(B) > 0$. *Then for any* $\epsilon > 0$, *there is* $m \in \mathbb{N}$ *such that, for all* $k \in \mathbb{N}$, *we have*

$$\underline{d}(\{x \in \mathbb{Z}^d \ : \ x + [-k, k] \subseteq A + B + [-m, m]\}) \geq \alpha - \epsilon.$$

2. *Suppose that* $d = 1$, $\underline{d}(A) = \alpha > 0$ *and* $\underline{d}(B) = \beta > 0$. *Then there is* $m \in \mathbb{N}$ *such that, for all* $k \in \mathbb{N}$, *we have*

$$\underline{d}(\{x \in \mathbb{Z} \ : \ x + [-k, k] \subseteq A + B + [-m, m]\}) \geq \min(\alpha + \beta, 1).$$

Question

Can we prove amenable group versions of these results?

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