Dynamical Cubes and a criteria for systems having product extensions

Joint work with Sebastián Donoso

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Systems with commuting transformations

- (X, S, T): topological dynamical system, $ST = TS$;
- (X, S, T) is minimal: $\{S^n T^m x \colon m, n \in \mathbb{Z}\}$ is dense in X for all $x \in X$.

Example

- $X = \{0, 1\}^{\mathbb{Z}^2}$;
- $\sigma_{(1,0)} \colon X \to X, (\sigma_{(1,0)}\omega)_{i,j} = \omega_{i+1,j}$;
- $\sigma_{(0,1)} \colon X \to X, (\sigma_{(0,1)}\omega)_{i,j} = \omega_{i,j+1}$.

$(X, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a topological system with

$$\sigma_{(1,0)} \circ \sigma_{(0,1)} = \sigma_{(1,0)} \circ \sigma_{(0,1)}.$$

Question

Let (X_1, T_1) and (X_2, T_2) be minimal topological dynamical systems. The system

$$(X_1 \times X_2, T_1 \times \mathrm{id}, \mathrm{id} \times T_2)$$

is the product of (X_1, T_1) and (X_2, T_2).

Question

How far is a system (X, S, T) from being a product system, or from being a factor of a product system?

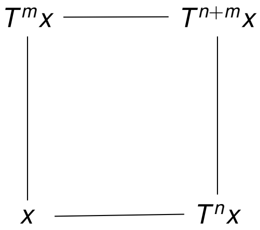
Cube structures for (X, T)

(X, T): minimal topological system.

Cube structure $\mathbf{Q}_T^{[2]}(X) \subseteq X^4$ (Host, Kra and Maass, 2010):

$$\mathbf{Q}_T^{[2]}(X) = \overline{\{(x, T^n x, T^m x, T^{n+m} x) : x \in X, n, m \in \mathbb{Z}\}}.$$

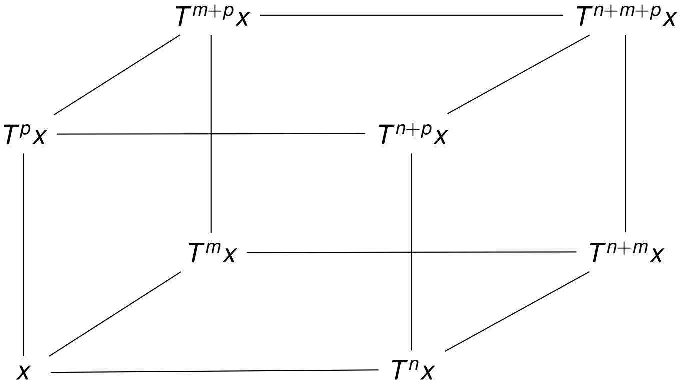
Every element of $\mathbf{Q}_T^{[2]}(X)$ is called a dynamical cube.



Cube structures for (X, T)

Cube structure $\mathbf{Q}_T^{[3]}(X) \subseteq X^8$:

$$\overline{\{(x, T^n x, T^m x, T^{n+m} x, T^p x, T^{n+p} x, T^{m+p} x, T^{n+m+p} x) : x \in X, n, m, p \in \mathbb{Z}\}}.$$



Example

(X, T): a rotation on a compact abelian group, $Tx = x + \alpha$.

$$
\begin{array}{ccc}
T^m x = x + m\alpha & \text{———} & T^{m+n} x = x + (n+m)\alpha \\
| & & | \\
x & \text{———} & T^n x = x + n\alpha
\end{array}
$$

$T^{m+n}x$ is determined by other coordinates: $T^{m+n}x = T^m x + T^n x - x$.

This property characterizes rotations on compact abelian groups.

Structure theorem for (X, T)

- G: d-step nilpotent group;
- $\Gamma \subset G$: discrete and cocompact;
- (X, T): is a nilsystem of order d if $X = G/\Gamma$ and $Tx = g \cdot x$ for some $g \in G$.

Theorem (Host, Kra, Maass, 2010)

The following are equivalent:

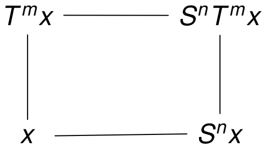
1. *The last coordinate of $\mathbf{Q}_T^{[d]}(X)$ is determined by the other $2^d - 1$ coordinates;*
2. *(X, T) is an inverse limit of nilsystems of order $d - 1$.*

Cube structures for (X, S, T)

(X, S, T): minimal topological system, $ST = TS$.

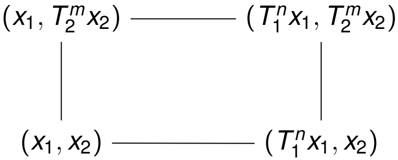
Cube Structure $\mathbf{Q}_{S,T}(X)$:

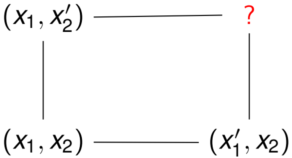
$$\mathbf{Q}_{S,T}(X) = \overline{\{(x, S^n x, T^m x, S^n T^m x) : x \in X, n, m \in \mathbb{Z}\}}.$$



Example 2

$(X_1 \times X_2, T_1 \times \mathrm{id}, \mathrm{id} \times T_2)$: a product system.





Example 2

$$\begin{array}{ccc} (x_1, x_2') & \text{———} & {\color{red}(x_1', x_2')} \\ | & & | \\ (x_1, x_2) & \text{———} & (x_1', x_2) \end{array}$$

For a product system, the last coordinate of dynamical cubes is determined by the previous ones.

What about the converse?

Main result: Characterizing product systems

Theorem (Host, Kra, Maass, 2010)

The following are equivalent:

1. *The last coordinate of* $\mathbf{Q}_T^{[2]}(X)$ *is determined by all other coordinates;*
2. (X, T) *is a rotation on a compact Abelian group.*

Theorem (Donoso, S.)

The following are equivalent:

1. *The last coordinate of* $\mathbf{Q}_{S,T}(X)$ *is determined by all other coordinates;*
2. (X, S, T) *is a factor of a product system.*

General structure for (X, T)

- (X, T): minimal topological system;
- $\mathcal{R}_T^{[2]} = \{(x, y) \in X \times X \colon (x, y, a, a) \in \mathbf{Q}_T^{[2]}$ for some $a \in X\}$.

Theorem (Host, Kra, Maass, 2010 (distal); Shao, Ye, 2012)

$\mathcal{R}_T^{[2]}$ is an equivalence relation, and $(X/\mathcal{R}_T^{[2]}, T)$ is the maximal equicontinuous factor of X.

$$(X, S, T)$$
$$\downarrow$$
equicontinuous $\quad (X/\mathcal{R}_T^{[2]}, T)$

Main result: General structure for (X, S, T)

- $(X, S, T)\colon ST = TS$;
- $\mathcal{R}_S = \{(x, y) \in X \times X \colon (x, y, a, a) \in \mathbf{Q}_{S,T} \text{ for some } a \in X\}$;
- $\mathcal{R}_T = \{(x, y) \in X \times X \colon (x, b, y, b) \in \mathbf{Q}_{S,T} \text{ for some } b \in X\}$;
- $\mathcal{R}_{S,T} = \mathcal{R}_S \cap \mathcal{R}_T$.

Theorem (Donoso, S.)

If $\mathcal{R}_{S,T}$ is an equivalence relation, then $(X/\mathcal{R}_{S,T}, S, T)$ is the maximal factor of X which has a product extension.

$$\begin{array}{ccc} & & (X, S, T) \\ & & \downarrow \\ (X_1 \times X_2, T_1 \times \mathrm{id}, \mathrm{id} \times T_2) & \longrightarrow & (X/\mathcal{R}_{S,T}, S, T) \end{array}$$

Application: computing the automorphism group

The automorphism group $Aut(X)$ of (X, T) is the collection of continuous maps $\phi\colon X \to X$ such that

$$T \circ \phi = \phi \circ T.$$

Example:

- (G, T) is a rotation on a compact Abelian group: $Aut(G) \cong G$;
- (X, T) is a d-step nilsystem: $Aut(X)$ is a d-step nilpotent group (Donoso, Durand, Maass, Petite, 2015);
- $(X, T) = (\{0, 1\}^{\mathbb{Z}}, \sigma)$: $Aut(X)$ is extremely large.

Robinson tiling

Consider the following set of tiles and their rotations and reflections:

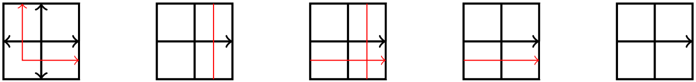
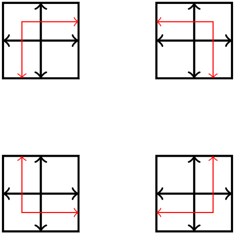


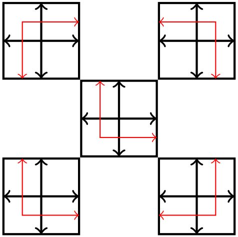
Figure : The Robinson Tiles. The first tile is called a cross

Let $\mathcal{A}_{Rob}$ denote this set of tiles. Then $\#\mathcal{A}_{Rob} = 28$.

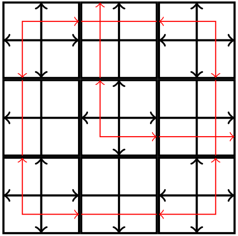
Robinson tiling

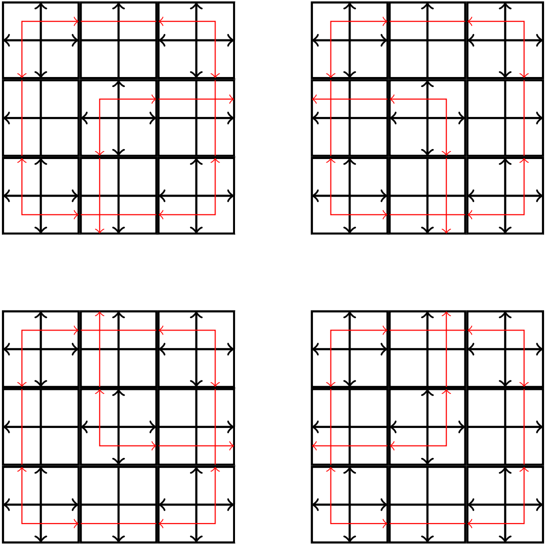


Robinson tiling

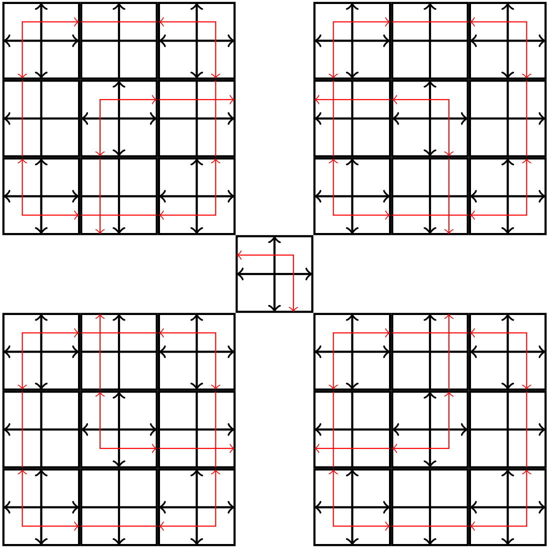


Robinson tiling

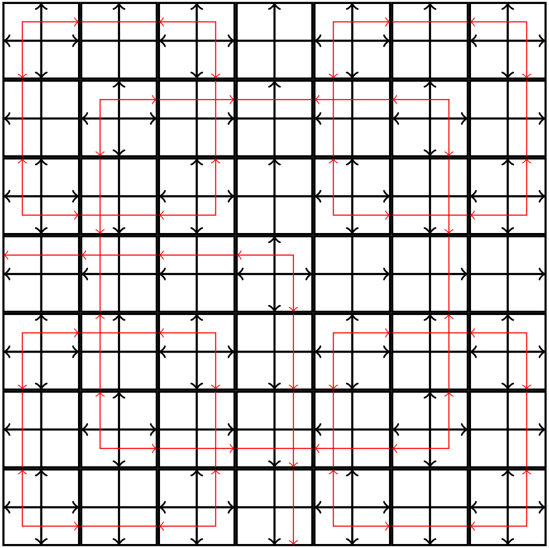




Robinson tiling



Robinson tiling



Automorphism group of Robinson tiling

We can continue this process to obtain a point $x_0 \in \mathcal{A}_{Rob}^{\mathbb{Z}^2}$.

X_{Rob}: the orbit closure of x_0 under $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$.

The automorphism group $Aut(X_{Rob})$ of $(X_{Rob}, \sigma_{(1,0)}, \sigma_{(0,1)})$ is the collection of continuous maps $\phi\colon X_{Rob} \to X_{Rob}$ such that

$$\sigma_{(1,0)} \circ \phi = \phi \circ \sigma_{(1,0)}$$

and

$$\sigma_{(0,1)} \circ \phi = \phi \circ \sigma_{(0,1)}.$$

Lemma

For all $\phi \in Aut(X_{Rob})$, $(x, y) \in \mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}$ iff $(\phi(x), \phi(y)) \in \mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}$.

Automorphism group of Robinson tiling

Proposition

There exists $A \subset X_{Rob}$ with $|A| = 28$ such that

- *For $x \in A$, $(x, y) \in \mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}$ iff $y \in A$;*
- *For all $x, y \in A$, there exist $m, n \in \mathbb{Z}$ such that $y = \sigma_{(1,0)}^m \sigma_{(0,1)}^n x$.*

Theorem

The automorphism group of $(X_{Rob}, \sigma_{(1,0)}, \sigma_{(0,1)})$ is spanned by $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$.

Proof:
Pick $\phi \in Aut(X_{Rob})$. By Lemma, $\phi(A) = A$.
Fix any $y \in A$, then there exist $m, n \in \mathbb{Z}$ such that $\phi(y) = \sigma_{(1,0)}^m \sigma_{(0,1)}^n y$.
Since $(X_{Rob}, \sigma_{(1,0)}, \sigma_{(0,1)})$ is minimal, $\phi(x) = \sigma_{(1,0)}^m \sigma_{(0,1)}^n x$ for all $x \in X$.

Other applications: pointwise multiple averages

See the next talk.

Thank you for your attention.

Theorem

Let (X, S, T) be a minimal system with commuting transformations S and T. The following are equivalent:

1. (X, S, T) *is a factor of a product system;*
2. *If* $\mathbf{x}$ *and* $\mathbf{y} \in \mathbf{Q}_{S,T}$ *have three coordinates in common, then* $\mathbf{x} = \mathbf{y}$*;*
3. *If* $(x, y, a, a) \in \mathbf{Q}_{S,T}$ *for some* $a \in X$*, then* $x = y$*; (*$\mathcal{R}_S = \Delta_X$*);*
4. *If* $(x, b, y, b) \in \mathbf{Q}_{S,T}$ *for some* $b \in X$*, then* $x = y$*; (*$\mathcal{R}_T = \Delta_X$*);*
5. *If* $(x, y, a, a) \in \mathbf{Q}_{S,T}$ *and* $(x, b, y, b) \in \mathbf{Q}_{S,T}$ *for some* $a, b \in X$*, then* $x = y$*. (*$\mathcal{R}_{S,T} = \Delta_X$*).*