Restriction theory and perturbations of Weyl sums.

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1. Introduction

Connection with additive combinatorics:

Theorem (Smith, 2009; Keil, 2014; Henriot, 2014)

Suppose that $s \geq 7$. Then there exists $c_s > 0$ with the following property. Whenever $\lambda_1, \ldots, \lambda_s \in \mathbb{Z} \setminus \{0\}$ satisfy $\lambda_1 + \ldots + \lambda_s = 0$, and $\mathcal{A} \subseteq \mathbb{N}$ satisfies

$$N^{-1} card(\mathcal{A} \cap [1, N]) \geq (\log N)^{-c_s} \quad (large\ N),$$

then there exist infinitely many non-trivial solutions $\mathbf{n} \in \mathcal{A}^s$ to the equation

$$\lambda_1 n_1^2 + \ldots + \lambda_s n_s^2 = 0.$$

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Analogous to Roth's theorem on 3-term arithmetic progressions, relating to the equation

$$n_1 - 2n_2 + n_3 = 0.$$

Idea is to add an extra equation to obtain the simultaneous equations

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This system of equations is translation-dilation invariant (just like $n_2 - 2n_2 + n_3 = 0$), and hence amenable to density concentration arguments in combination with suitable ideas from Fourier analysis (such as the Hardy-Littlewood (circle) method).

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Key ingredient: Estimates for mean values of the shape

$$\int_{[0,1)^2} \left| \sum_{1\le n\le X} a_n e(\alpha_1 n + \alpha_2 n^2) \right|^s \mathrm{d}\boldsymbol{\alpha} \quad (s > 6),$$

in which (a_n) is a sequence of real numbers.

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Would like to generalise this all to higher degree problems.

2. Restriction theory (Bourgain, 1993):

Suppose that (a_n) is a sequence of complex numbers. Then:

$$\oint \Big| \sum_{1\le n\le X} a_n e(n^2\alpha + n\beta)\Big|^{2s} \mathrm{d}\alpha\,\mathrm{d}\beta \ll \Big(\sum_{n\le X} |a_n|^2 \Big)^s \quad (s<3),$$

$$\oint \Big| \sum_{1\le n\le X} a_n e(n^2\alpha + n\beta)\Big|^{6} \mathrm{d}\alpha\,\mathrm{d}\beta \ll X^{\varepsilon} \Big(\sum_{n\le X} |a_n|^2 \Big)^3,$$

$$\oint \Big| \sum_{1\le n\le X} a_n e(n^2\alpha + n\beta)\Big|^{2s} \mathrm{d}\alpha\,\mathrm{d}\beta \ll X^{s-3} \Big(\sum_{n\le X} |a_n|^2 \Big)^s \quad (s>3).$$

Sketch proof:

By orthogonality, the integral

$$\oint \Big| \sum_{1 \le n \le X} a_n e(n^2\alpha + n\beta) \Big|^6 \, \mathrm{d}\alpha \, \mathrm{d}\beta$$

counts the number of solutions of the simultaneous equations

$$\left.\begin{aligned} n_1^2 + n_2^2 + n_3^2 &= n_4^2 + n_5^2 + n_6^2 \\ n_1 + n_2 + n_3 &= n_4 + n_5 + n_6 \end{aligned}\right\} ,$$

with each solution counted with weight

$$a_{n_1} a_{n_2} a_{n_3} \overline{a}_{n_4} \overline{a}_{n_5} \overline{a}_{n_6} .$$

Sketch proof for the case $k = 2$ and $s = 3$:

By orthogonality, the integral

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counts the number of solutions of the simultaneous equations

$$\left.\begin{aligned} n_1^2 + n_2^2 - n_3^2 &= n_4^2 + n_5^2 - n_6^2 \\ n_1 + n_2 - n_3 &= n_4 + n_5 - n_6 \end{aligned}\right\},$$

with each solution counted with weight

$$a_{n_1} a_{n_2} \overline{a}_{n_3} \overline{a}_{n_4} \overline{a}_{n_5} a_{n_6}.$$

Let $\mathcal{B}(\mathbf{h})$ denote the set of integral solutions of the equation

$$\left.\begin{aligned} n_1^2 + n_2^2 - n_3^2 &= h_2 \\ n_1 + n_2 - n_3 &= h_1 \end{aligned}\right\},$$

with $1 \le n_i \le X$.

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with $1 \le n_i \le X$.
Then by Cauchy's inequality,

$$\oint \Bigl| \sum_{1 \le n \le X} a_n e(n^2\alpha + n\beta) \Bigr|^6 \, \mathrm{d}\alpha \, \mathrm{d}\beta = \sum_{|h_i| \le 2X^i \, (i=1,2)} \Bigl(\sum_{(n_1,n_2,n_3) \in \mathcal{B}(\mathbf{h})} a_{n_1} a_{n_2} \overline{a}_{n_3} \Bigr)^2$$
$$\le \sum_{\mathbf{h}} \sum_{n_1,n_2,n_3} |\mathcal{B}(\mathbf{h})| |a_{n_1} a_{n_2} a_{n_3}|^2.$$

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Then by Cauchy's inequality,

$$\begin{aligned} \oint \Bigl| \sum_{1 \le n \le X} a_n e(n^2\alpha + n\beta) \Bigr|^6 \, \mathrm{d}\alpha \, \mathrm{d}\beta &= \sum_{|h_i| \le 2X^i \, (i=1,2)} \Bigl(\sum_{(n_1,n_2,n_3) \in \mathcal{B}(\mathbf{h})} a_{n_1} a_{n_2} \overline{a}_{n_3} \Bigr)^2 \\ &\le \sum_{\mathbf{h}} \sum_{n_1,n_2,n_3} |\mathcal{B}(\mathbf{h})| |a_{n_1} a_{n_2} a_{n_3}|^2. \end{aligned}$$

But $|\mathcal{B}(\mathbf{h})|$ is bounded above by the number of solutions of

$$\begin{aligned} h_1^2 - h_2 &= (n_1 + n_2 - n_3)^2 - (n_1^2 + n_2^2 - n_3^2) \\ &= 2(n_1 - n_3)(n_2 - n_3), \end{aligned}$$

and this is $O(X^\varepsilon)$ unless $n_1 = n_3$ or $n_2 = n_3$.

One should remove the special solutions with $n_1 = n_3$ or $n_2 = n_3$ in advance, and for the remaining solutions one finds that

$$\oint \Big| \sum_{1 \le n \le X} a_n e(n^2\alpha + n\beta) \Big|^6 \mathrm{d}\alpha \, \mathrm{d}\beta \ll X^\varepsilon \sum_{n_1, n_2, n_3} |a_{n_1} a_{n_2} a_{n_3}|^2$$

$$\ll X^\varepsilon \Big(\sum_n |a_n|^2 \Big)^3.$$

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$$\ll X^\varepsilon \Big(\sum_n |a_n|^2 \Big)^3.$$

Key observation: With $\mathcal{B}(\mathbf{h})$ the set of integral solutions of the equation

$$\left. \begin{aligned} n_1^2 + n_2^2 - n_3^2 &= h_2 \\ n_1 + n_2 - n_3 &= h_1 \end{aligned} \right\},$$

with $1 \le n_i \le X$, one has $|\mathcal{B}(\mathbf{h})| \ll X^\varepsilon$ (Very strong control of the number of solutions of the associated Diophantine system).

The main conjecture:

Let $t \geq 1$ and $1 \leq k_1 < k_2 < \ldots < k_t = k$ be integers, and put

$$K = k_1 + \ldots + k_t.$$

Consider a sequence $(a_n)_{n=1}^{\infty}$ of complex numbers, not all zero, and define

$$f_{k,\mathbf{a}}(\boldsymbol{\alpha}; X) = \sum_{1 \leq n \leq X} a_n e(n^{k_1}\alpha_1 + \ldots + n^{k_t}\alpha_t).$$

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Conjecture (Main Restriction Conjecture)

For each $\varepsilon > 0$, one has

$$\oint |f_{\mathbf{k},\mathbf{a}}(\boldsymbol{\alpha}; X)|^{2s} \, \mathrm{d}\boldsymbol{\alpha} \ll \begin{cases} X^{\varepsilon} \left(\sum_{n \leq X} |a_n|^2 \right)^s, & \text{when } s \leq K, \\ X^{s-K} \left(\sum_{n \leq X} |a_n|^2 \right)^s, & \text{when } s > K. \end{cases}$$

Here, we write $\oint$ for $\int_{[0,1)^t}$.

Some observations, I: Special case $\mathbf{k} = (1, \ldots, k)$

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Consider the sequence $(a_n) = 1$. Then MRC implies that

$$\oint |f_{k,\mathbf{1}}(\boldsymbol{\alpha}; X)|^{2s} \, \mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}(X^s + X^{2s - \frac{1}{2}k(k+1)}),$$

an assertion equivalent to the Main Conjecture in Vinogradov's Mean Value Theorem.

$$f_{k,\mathbf{a}}(\boldsymbol{\alpha};X)=\sum_{1\le n\le X} a_n e(n^{k_1}\alpha_1+\ldots+n^{k_t}\alpha_t).$$

Theorem (Bourgain, 1993; Bourgain-Demeter, 2014; Hughes, 201?)

For each $\varepsilon>0$, one has MRC in the shape

$$\oint |f_{\mathbf{k},\mathbf{a}}(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}\ll X^{\varepsilon}(1+X^{s-K})\left(\sum_{n\le X}|a_n|^2\right)^s$$

whenever:
(a) $\mathbf{k}=(1,2)$, *or*
(b) $s\le 2t-1$, *or*
(c) $s\ge 2k(k-1)$.
Moroever, the factor X^{ε} may be removed when $s>2k(k-1)$.

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The result (c) and its sequel depends on the latest "efficient congruencing" results in Vinogradov's mean value theorem (W., 2014).

3. Efficient congruencing

Recent techniques applied in the context of Vinogradov's mean value theorem allow one to establish:

Theorem (W. 2015)

For each $\varepsilon > 0$, one has MRC in the shape

$$\oint |f_{k,\mathbf{a}}(\boldsymbol{\alpha}; X)|^{2s}\, \mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}(1 + X^{s-K}) \left(\sum_{n \le X} |a_n|^2 \right)^s$$

whenever:

(a) $\mathbf{k} = (1,2)$ *or* $(1,2,3)$ *(cf. classical* $\mathbf{k} = (1,2)$*), or*

(b) $1 \le s \le D(t)$*, where* $D(2) = 3$, $D(3) = 6$, $D(4) = 9$, $D(5) = 14$, ... , *and* $D(t) = \frac{1}{2}t(t+1) - \frac{1}{3}t + O(t^{2/3})$ *(cf. Bourgain-Demeter* $D(t) = 2t - 1$*), or*

(c) $s \ge k(k-1)$ *(cf. classical* $s \ge 2k(k-1)$*).*

Moroever, the factor X^{ε} *may be removed when* $s > k(k-1)$*.*

We now aim to sketch the ideas underlying a slightly simpler result:

Theorem

For each $\varepsilon > 0$, one has MRC in the shape

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whenever $s \leq t^2/4$.

It is worth noting that we tackle the mean value directly, rather than using results about Vinogradov's mean value theorem (the special case $(a_n) = (1)$) indirectly.

Cartoon of strategy:

(1) Suppose that the mean value in question is significantly larger than we "expect" it to be. Say that, for some $\Lambda > 0$, one has

$$\oint |f_{\mathbf{k},\mathbf{a}}(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \gg X^{\Lambda}(1+X^{s-K})\left(\sum_{n\le X}|a_n|^2\right)^s.$$

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(2) Use arithmetic/harmonic analysis to show that, for some $\rho > 0$, a sequence of related mean values $K_n(X)$ satisfy an amplification property

$$K_{n+1}(X) \gg X^{\Lambda(1+\rho)} \times \text{"expected size" of } K_n(X).$$

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(3) Iterate, and show that for large enough N, the relation

$$\textit{upper bound for } K_N(X) \gg K_N(X) \gg X^{\Lambda(1+\rho)^N} \times \text{"expected" size of } K_N(X).$$

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upper bound for $K_N(X) \gg K_N(X) \gg X^{\Lambda(1+\rho)^N} \times$ *"expected" size of* $K_N(X)$.

(4) Obtain a contradiction, and infer that $\Lambda \leq 0$. (cf. density increment strategies).

Consider an auxiliary prime number p (for now, think of p as being a very small power of X).

Write

$$\rho_c(\xi) = \rho_c(\xi; \mathbf{a}) = \Bigg(\sum_{\substack{1 \le n \le X \\ n \equiv \xi \pmod{p^c}}} |a_n|^2 \Bigg)^{1/2},$$

and then define

$$\widetilde{f}_{\mathbf{a}}(\boldsymbol{\alpha}; X) = \rho_0(1)^{-1} \sum_{1 \le n \le X} a_n e(n^{k_1}\alpha_1 + \ldots + n^{k_t}\alpha_t).$$

[Note: if $a_n = 0$ for all n, then define $\widetilde{f}_{\mathbf{a}} = 0$.]

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We investigate

$$U_{s,\mathbf{k}}(X; \mathbf{a}) = \oint |\widetilde{f}_{\mathbf{a}}(\boldsymbol{\alpha}; X)|^{2s} \, \mathrm{d}\boldsymbol{\alpha}.$$

Observe that by Cauchy's inequality, one has

$$\begin{aligned}|f_{\mathbf{a}}(\boldsymbol{\alpha};X)| &= \Big| \sum_{1\le n\le X} a_n e(n^{k_1}\alpha_1+\ldots+n^{k_t}\alpha_t)\Big| \\ &\le X^{1/2}\Big(\sum_{n\le X} |a_n|^2\Big)^{1/2},\end{aligned}$$

whence

$$|\widetilde{f}_{\mathbf{a}}(\boldsymbol{\alpha};X)| \le X^{1/2}.$$

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Thus

$$U_{s,\mathbf{k}}(X;\mathbf{a})=\oint |\widetilde{f}_{\mathbf{a}}(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}\ll X^s.$$

Moreover, one has that $U_{s,\mathbf{k}}(X;\mathbf{a})$ is scale-invariant, by which we mean that it is invariant on scaling (a_n) to (γa_n) for any $\gamma>0$.

Define

$$\lambda_s = \limsup_{X\to\infty} \sup_{\substack{(a_n)\in\mathbb{C}^{[X]}\\ |a_n|\le 1}} \frac{\log U_{s,\mathbf{k}}(X;\mathbf{a})}{\log X}.$$

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Then there exists a sequence $(X_m)_{m=1}^{\infty}$ with $\lim_{m\to\infty} X_m = +\infty$ such that, for some sequence $(a_n)\in\mathbb{C}^{[X_m]}$ with $|a_n|\le 1$, one has that for each $\varepsilon>0$,

$$U_{s,\mathbf{k}}(X_m;\mathbf{a}) \gg X^{\lambda_s-\varepsilon},$$

whilst whenever $X_m^{1/2}\le Y\le X_m$, and for all sequences (a_n), at the same time one has

$$U_{s,\mathbf{k}}(Y;\mathbf{a}) \ll Y^{\lambda_s+\varepsilon}.$$

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We now fix such a value $X = X_m$ sufficiently large, and abbreviate

$$\Lambda = \lambda_{s+r},$$

where $r \le t/2$.

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Approach this problem through an auxiliary mean value. Define

$$\mathfrak{f}_c(\boldsymbol{\alpha}; \xi) = \rho_c(\xi)^{-1} \sum_{\substack{1 \le n \le X \\ n \equiv \xi \pmod{p^c}}} a_n e(n^{k_1}\alpha_1 + \ldots + n^{k_t}\alpha_t),$$

and then put

$$K_{a,b}(X) = \rho_0(1)^{-4} \sum_{\xi=1}^{p^a} \sum_{\eta=1}^{p^b} \rho_a(\xi)^2 \rho_b(\eta)^2 \oint |\mathfrak{f}_a(\boldsymbol{\alpha}; \xi)^{2r} \mathfrak{f}_b(\boldsymbol{\alpha}; \eta)^{2s}| \, \mathrm{d}\boldsymbol{\alpha}.$$

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Strategy:

(i) Show that if $U_{s+r,\mathbf{k}}(X;\mathbf{a}) \gg X^{\Lambda}$, then $K_{0,1}(X) \gg X^{\Lambda}$.

(ii) Show that whenever $K_{a,b}(X) \gg X^{\Lambda}(p^{\psi})^{\Lambda}$, then there is a small non-negative integer h with the property that

$$K_{a',b'}(X) \gg X^{\Lambda}(p^{\psi'})^{\Lambda},$$

where, with $m = t - r + 1$,

$$\psi' = m\psi + (m-1)b, \quad a' = b, \quad b' = mb + h.$$

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By iterating this process, we obtain sequences $(a^{(n)})$, $(b^{(n)})$, $(\psi^{(n)})$ with

$$b^{(n)} \approx m^n \quad \text{and} \quad \psi^{(n)} \approx nm^n$$

for which

$$K_{a^{(n)},b^{(n)}}(X) \gg X^{\Lambda}(p^{\psi^{(n)}})^{\Lambda}.$$

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Suppose that $\Lambda > 0$ and $s/r \leq m$. Then the right hand side here increases so rapidly that, for large enough values of n, it is larger than the trivial estimate for the left hand side. This gives a contradiction, so that $\Lambda \leq 0$.

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Upshot: Provided that $s \le mr = (t - r + 1)r$, then

$$U_{s+r,\mathbf{k}}(X; \mathbf{a}) \ll X^\varepsilon.$$

Take $r = [t/2]$, so that we can take $s + r$ essentially to be $t^2/4$. Then

$$\oint |f_{\mathbf{a}}(\boldsymbol{\alpha}; X)|^{2s+2r} \, \mathrm{d}\boldsymbol{\alpha} \ll X^\varepsilon \left(\sum_{1 \le n \le X} |a_n|^2 \right)^s.$$

4. Translation invariance, and the congruencing idea

Observe that the system of equations

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = 0 \quad (1 \le j \le t) \tag{1}$$

has a solution $\mathbf{x}, \mathbf{y}$ if and only if, for any integral shift a, the system of equations

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To see this, note that

$$\sum_{l=1}^{j} \binom{j}{l} a^{j-l} \sum_{i=1}^{s} ((x_i - a)^l - (y_i - a)^l) = \sum_{i=1}^{s} ((x_i - a + a)^j - (y_i - a + a)^j).$$

The mean value

$$\oint |\mathfrak{f}_a(\boldsymbol{\alpha};\xi)^{2r}\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s}|\,\mathrm{d}\boldsymbol{\alpha}$$

counts (with weights) the number of integral solutions of the system

$$\sum_{i=1}^{r}(x_i^{k_j}-y_i^{k_j})=\sum_{l=1}^{s}((p^bu_l+\eta)^{k_j}-(p^bv_l+\eta)^{k_j})\quad (1\le j\le t),$$

with $1 \le \mathbf{x}, \mathbf{y} \le X$ and $(1-\eta)/p^b \le \mathbf{u}, \mathbf{v} \le (X-\eta)/p^b$.

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By taking suitable linear combinations of equations, we obtain

$$\sum_{i=1}^r (\Psi_j(x_i)-\Psi_j(y_i)^j) \equiv p^{jb}\sum_{l=1}^s (u_l^j-v_l^j) \pmod{p^{tb}} \quad (1\le j\le t),$$

for suitable (independent) polynomials $\Psi_j(u)$ of degree at most k_t, depending on η and j (and $k_1,\ldots,k_t$).

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In this way, we obtain a system of congruence conditions modulo p^{mb} for $t-r+1\le j\le t$.

$$\sum_{i=1}^{r} \Psi_j(x_i) \equiv \sum_{i=1}^{r} \Psi_j(y_i) \pmod{p^{mb}} \quad (t-r+1 \leq j \leq t).$$

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By linear algebra, this takes the shape

$$\sum_{i=1}^{r} (p^a w_i)^j + O_{\mathbf{w}}(p^{ta}) \equiv \sum_{i=1}^{r} (p^a z_i)^j + O_{\mathbf{z}}(p^{ta}) \pmod{p^{mb}} \quad (1 \leq j \leq r).$$

Suppose that $\mathbf{x}$ is *well-conditioned*, by which we mean that $x_1, \ldots, x_t$ lie in distinct congruence classes modulo p. Then, given an integral t-tuple $\mathbf{n}$, the solutions of the system

$$\sum_{i=1}^{t} \Psi_j(x_i) \equiv n_j \pmod{p} \quad (t-r+1 \leq j \leq t),$$

with $1 \leq \mathbf{x} \leq p$, may be lifted (essentially) uniquely to solutions of

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Now we are counting solutions with weights, so we reinsert this congruence information back into the mean value $K_{a,b}(X)$ to obtain the relation

$$K_{a,b}(X) \ll \rho_0(1)^{-4} \sum_{\xi=1}^{p^a} \sum_{\eta=1}^{p^b} \rho_a(\xi)^2 \rho_b(\eta)^2 \Xi,$$

where

$$\Xi = \oint \left(\sum_{\substack{1 \le \xi' \le p^{mb} \\ \xi' \equiv \xi \pmod{p^a}}} \frac{\rho_{mb}(\xi')^2}{\rho_a(\xi)^2} |\mathfrak{f}_{mb}(\boldsymbol{\alpha}; \xi')|^2 \right)^r |\mathfrak{f}_b(\boldsymbol{\alpha}; \eta)|^{2s} \, \mathrm{d}\boldsymbol{\alpha}.$$

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But by Hölder's inequality, the term here raised to power r is bounded above by

$$\frac{1}{\rho_a(\xi)^{2r}} \left(\sum_{\substack{1 \le \xi' \le p^{mb} \\ \xi' \equiv \xi \pmod{p^a}}} \rho_{mb}(\xi')^2 |\mathfrak{f}_{mb}(\boldsymbol{\alpha}; \xi')|^{2s} \right)^{\frac{r}{s}} \left(\sum_{\substack{1 \le \xi' \le p^{mb} \\ \xi' \equiv \xi \pmod{p^a}}} \rho_{mb}(\xi')^2 \right)^{r - \frac{r}{s}}$$

$$\ll \left(\rho_a(\xi)^{-2} \sum_{\substack{1 \le \xi' \le p^{mb} \\ \xi' \equiv \xi \pmod{p^a}}} \rho_{mb}(\xi')^2 |\mathfrak{f}_{mb}(\boldsymbol{\alpha}; \xi')|^{2s} \right)^{r/s}.$$

Then another application of Hölder's inequality yields

$$\begin{aligned}\Xi &\ll \oint \left(\rho_a(\xi)^{-2} \sum_{\xi'} \rho_{mb}(\xi')^2 |\mathfrak{f}_{mb}(\boldsymbol{\alpha};\xi')|^{2s} \right)^{r/s} |\mathfrak{f}_b(\boldsymbol{\alpha};\eta)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \\ &\ll \Xi_1^{r/s}\Xi_2^{1-r/s},\end{aligned}$$

where

$$\Xi_1 = \rho_a(\xi)^{-2} \sum_{\xi'} \rho_{mb}(\xi')^2 \oint |\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2r} \mathfrak{f}_{mb}(\boldsymbol{\alpha};\xi')^{2s}|\,\mathrm{d}\boldsymbol{\alpha}$$

and

$$\Xi_2 = \oint |\mathfrak{f}_b(\boldsymbol{\alpha};\eta)|^{2s+2r}\,\mathrm{d}\boldsymbol{\alpha}.$$

Recall that

$$K_{a,b}(X) \ll \rho_0(1)^{-4} \sum_{\xi=1}^{p^a} \sum_{\eta=1}^{p^b} \rho_a(\xi)^2 \rho_b(\eta)^2 \Xi,$$

From here, yet another application of Hölder's inequality gives

$$K_{a,b}(X) \ll \Xi_3^{r/s} \Xi_4^{1-r/s},$$

where

$$\Xi_3 = \rho_0(1)^{-4} \sum_{\eta=1}^{p^b} \sum_{\xi'=1}^{p^{mb}} \rho_b(\eta)^2 \rho_{mb}(\xi')^2 \oint |\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2r} \mathfrak{f}_{mb}(\boldsymbol{\alpha};\xi')^{2s}| \, \mathrm{d}\boldsymbol{\alpha},$$

and (but note inhomogeneity!)

$$\begin{aligned} \Xi_4 &= \rho_0(1)^{-4} \sum_{\eta=1}^{p^b} \sum_{\xi=1}^{p^a} \rho_b(\eta)^2 \rho_a(\xi)^2 \oint |\mathfrak{f}_b(\boldsymbol{\alpha};\eta)|^{2s+2r} \, \mathrm{d}\boldsymbol{\alpha} \\ &\ll (X/M^b)^{\Lambda+\varepsilon}. \end{aligned}$$

Then one can check that

$$[[K_{a,b}(X)]] \ll [[K_{b,mb}(X)]]^{r/s}(X/M^b)^{(1-r/s)(\Lambda+\varepsilon)}.$$

Given the hypothesis that

$$[[K_{a,b}(X)]] \gg X^\Lambda (p^\psi)^\Lambda,$$

this implies that

$$[[K_{b,mb}(X)]] \gg X^\Lambda (p^{\psi'})^\Lambda,$$

where

$$\psi' = m\psi + (m-1)b,$$

which is a little stronger than we had claimed earlier. Then the "cartoon" argument analogous to density increment method applies, and we are done.

Final Comments:

One should now be able to analyse the solubility of systems of equations

$$\sum_{i=1}^{s} \lambda_i x_i^{k_j} = 0 \quad (1 \le j \le t),$$

subject to $\lambda_1 + \ldots + \lambda_s = 0$, in dense sets $\mathcal{A} \subseteq \mathbb{N}$.

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Applications to upper bounds for exponential sums:

$$\sum_{1 \le n \le X} a_n e(\alpha_1 x^{k_1} + \ldots + \alpha_t x^{k_t}) \ll X^{1/2+O(1/t)}$$

on sets of $(\alpha_1, \ldots, \alpha_{t-1}) \in [0,1]^{t-1}$ of measure 1.

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Thanks for your attention, and to the organisers!