Small maximal independent sets

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Table of Contents

Ramsey's theorem (for 2 colors)

Theorem (Ramsey)

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- $R(3,3)$: least integer N for which each blue-red edge coloring on K_N contains either a red or a blue triangle.
- $R(3,3) \leq 6$: Theorem on friends and strangers.
- $R(3,3) > 5$: Pentagon with red edges, then color "inside" edges blue.

The probabilistic method (Erdős)

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- If $\mathbb{E}(X) < 1$ then a non-monochromatic example exists, so $R(r,r) \geq 2^{r/2}$.
- Can one explicitly (pol. time algorithm in nr. of vertices) construct for some fixed $\epsilon > 0$ a 2-edge coloring of the complete graph on $N > (1+\epsilon)^n$ vertices with no monochromatic clique of size n?

Main Result

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Let $\delta, \varepsilon \in \mathbb{R}^+$ and let G be a v-vertex d-regular δ-sparse graph. If d is large enough relative to δ and ε, then G contains a maximal independent set of size at most

$$\frac{(1+\varepsilon)v \log d}{d}.$$

Table of Contents

The classical generalized quadrangles

- non-singular quadric of Witt index 2 in $\mathrm{PG}(3, q)$ ($O^{+}(4, q)$), $\mathrm{PG}(4, q)$ ($O(5, q)$) and $\mathrm{PG}(5, q)$ ($O^{-}(6, q)$).

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- Symplectic quadrangle $W(q)$, of order q ($\mathrm{Sp}(4, q)$).
- Not all GQs are classical (e.g. Tits, Kantor, Payne).

Small maximal partial ovoids in GQs

$\mathcal{Q}$	Previous range for $\gamma(\mathcal{Q})$	Theorem	Ref.
$Q^-(5,q)$	$[2q, q^2/2]$	$[2q, 3q \log q]$	[DBKMS,EH,MS]
$Q(4,q)$, q odd	$[1.419q, q^2]$	$[1.419q, 2q \log q]$	[CDWFS,DBKMS]
$H(4,q^2)$	$[q^2, q^5]$	$[q^2, 5q^2 \log q]$	[MS]
$DH(4,q^2)$	$[q^3, q^5]$	$[q^3, 5q^3 \log q]$	/
$H(3,q^2)$, q odd	$[q^2, 2q^2 \log q]$	$[q^2, 3q^2 \log q]$	[AEL,M]

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- *ovoid* : set of points, no two of which are collinear.
- Main theorem: any GQ of order (s,t) has a maximal partial ovoid of size roughly $s\log(st)$.

Small maximal partial ovoids in polar spaces

Q	Known prior	Range from MT	Ref.
$Q(2n,q)$, q odd	$[q,q^n]$	$[q,(2n-2)q\log q]$	[BKMS]
$Q(2n,q)$, q even	$=q+1$		[BKMS]
$Q^+(2n+1,q)$	$[2q,q^n], n\geq 3$	$[2q,(2n-1)q\log q]$	[BKMS]
$Q^-(2n+1,q)$	$[2q,\frac{1}{2}q^{n+1}], n\geq 3$	$[2q,(2n-1)q\log q]$	[BKMS]
$W(2n+1,q)$	$=q+1$		[BKMS]
$H(2n,q^2)$	$[q^2,q^{2n+1}], n\geq 3$	$[q^2,(4n-3)q^2\log q]$	[JDBKL]
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- Small maximal partial spreads in polar spaces.
- Blocking circles in Möbius planes.
- Maximal partial spreads in projective space $\mathrm{PG}(n, q), n \geq 3$.
- For the latter: vertices=lines, edges=intersecting lines.
- δ-sparse system with $v = q^{2n-2}$, $d = q^n$, so maximal partial spread of size $(n-2)q^{n-2} \log q$.

Problem: How to prove lower bounds?

Theorem (Weil)

Let ξ be a character of $\mathbb{F}_q$ of order s. Let $f(x)$ be a polynomial of degree d over $\mathbb{F}_q$ such that $f(x) \neq c(h(x))^s$, where $c \in \mathbb{F}_q$. Then

$$|\sum_{a \in \mathbb{F}_q} \xi(f(a))| \leq (d-1)\sqrt{q}.$$

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- Gács and Szőnyi: In a Miquelian $3-(q^2+1, q+1, 1)$-design, the minimal number of circles through a given point needed to block all circles is always at least of order $\frac{1}{2}\log q$ using Weil's theorem.

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- This case involves estimates of quadratic character sums, becomes too complicated for other examples.
- Moreover many problems do not have an algebraic description.

Table of Contents

A technical condition for GQs

A GQ of order (s, t) is called *locally sparse* if for any set of three points, the number of points collinear with all three points is at most $s + 1$.

- Any GQ of order (s, s^2) is locally sparse (Bose-Shrikhande, Cameron)

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- Any GQ of order (s, s^2) is locally sparse (Bose-Shrikhande, Cameron)
- In particular, $Q^-(5, q)$ is locally sparse.
- $H(4, q^2)$ is **not** locally sparse.

A weaker theorem for GQs

Theorem

For any $\alpha > 4$*, there exists* $s_{\circ}(\alpha)$ *such that if* $s \geq s_{\circ}(\alpha)$ *and* $t \geq s(\log s)^{2\alpha}$*, then any locally sparse generalized quadrangle of order* (s, t) *has a maximal partial ovoid of size at most* $s(\log s)^{\alpha}$*.*

First round

- Fix a point $x \in \mathcal{P}$ and for each line l through x independently flip a coin with heads probability $ps = \frac{s \log t - \alpha s \log \log s}{t}$, where $\alpha > 4$.

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- On each line l where the coin turned up heads, select uniformly a point of $l \setminus \{x\}$ and denote the set of selected points by S.
- $U = \mathcal{P} \setminus (S \cup \{x\})^{\bowtie}$ (uncovered points not collinear with x).

Second round

Let $x^\star \in x^\perp \setminus S^{\bowtie}$. On each line $l \in \mathcal{L}$ through $x^\star$ with $l \cap U \neq \emptyset$, uniformly and randomly select a point of $l \cap U$. Moreover select a point x^+ on the line M through $x^\star$ and x different from x, and call this set of selected points T. Then clearly $S \cup T \cup \{x^+\}$ is a partial ovoid. So we will need to show that $S \cup T \cup \{x^+\}$ is maximal, and small.

A form of the Chernoff bound

A sum of independent random variables is concentrated according to the so-called Chernoff Bound. We shall use the Chernoff Bound in the following form. We write $X \sim \text{Bin}(n, p)$ to denote a binomial random variable with probability p over n trials.

Proposition

Let $X \sim Bin(n, p)$. Then for $\delta \in [0, 1]$,

$$\mathbb{P}(|X - pn| \geq \delta pn) \leq 2e^{-\delta^2 pn/2}.$$

Proof for GQs i

First we show $|S| \lesssim s \log t$ using the Chernoff Bound. There are $t+1$ lines through x, and we independently selected each line with probability ps and then one point on each selected line. So $|S| \sim \text{Bin}(t+1, ps)$ and $\mathbb{E}(|S|) = ps(t+1) \sim s \log t$. By Chernoff, for any $\delta > 0$,

$$\mathbb{P}(|S| \geq (1+\delta)s \log t) \leq 2\exp(-\tfrac{1}{2}\delta^2 s \log t) \to 0.$$

Therefore a.a.s. $|S| \lesssim s \log t$.

Three key properties

We can show that in selecting S, Properties I – III described below occur simultaneously a.a.s. as $s \to \infty$:

I. *For all lines* $\ell \in \mathcal{L}$ *disjoint from* x*,* $|\ell \cap U| < \lceil \log s \rceil$*.*
II. *For all* $u \in x^{\perp} \backslash S$*,* $|u^{\perp} \cap U| \lesssim s(\log s)^{\alpha}$
III. *For* $v, w \notin S \cup \{x\}; v \not\sim w$*,* $|\{v, w\}^{\perp} \cap U| \gtrsim (\log s)^{\alpha}$*.*

Proof for GQs ii

Assuming that a.a.s., S satisfies Properties I – III, we fix an instance of such a partial ovoid S with $|S| \lesssim s \log t$ and let T be as before. By Property II, $|T| \leq X_{x^*} \lesssim s(\log s)^\alpha$. Therefore

$$|S \cup T| \leq |S| + X_{x^*} + 1 \lesssim s \log t + s(\log s)^\alpha \lesssim s(\log s)^\alpha$$

Proof for GQs iii

For $v \in (x^\perp \setminus S^{\bowtie}) \cup U$ not collinear with x^*, a.a.s., $X_{vx^*} \geq \frac{1}{2}(\log s)^\alpha$ by Property III. By Property I, the probability that v is not collinear with any point in T is at most

$$\left(\frac{\log s - 1}{\log s}\right)^{X_{vx^*}} \leq \left(1 - \frac{1}{\log s}\right)^{\frac{1}{2}(\log s)^\alpha} \leq e^{-\frac{1}{2}(\log s)^3} < \frac{1}{s^5}$$

since $\alpha > 4$. Hence the expected number of points in $(x^\perp \setminus S^{\bowtie}) \cup U$ not collinear with any point in T is at most

$$s^{-5}|\mathcal{P}| \lesssim \frac{1}{s}.$$

It follows that a.a.s.,

$$(x^\perp \setminus (S^{\bowtie} \cup M)) \cup U \subset T^{\bowtie}$$

hence $S \cup T \cup \{x^+\}$ is a maximal partial ovoid.

Practical implementation

The randomized algorithm in this paper could be implemented, and we believe it is effective in finding maximal partial ovoids even in (s, t)-quadrangles where s is not too large. In addition, it can be deduced from the proof that the probability that the algorithm does not return a maximal partial ovoid of size at most $s(\log s)^{\alpha}, \alpha > 4$, is at most $s^{-\log s}$ if s is large enough.

Table of Contents

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- A *maximal independent set* in a set system $\mathcal{S}$ is a set I of atoms containing no block but such that the addition of any atom to I results in a set containing some block of $\mathcal{S}$.
- Let $\gamma_\circ(\mathcal{S})$ denote the smallest possible size of a maximal independent set in $\mathcal{S}$.

What do we want to prove?

For $\delta > 0$, an (n, d, r)-system $\mathcal{S}$ is *locally δ-sparse* if for $k \in \{1, 2\}$ and each pair of atoms x, y of $\mathcal{S}$, the maximum number of chains of length k with ends x and y is at most $\lceil d^{k-\frac{1}{r-1}-\delta} \rceil$.

Aim

Let $r \in \mathbb{Z}_+$ and $\delta \in \mathbb{R}_+$, and let $\mathcal{S}$ be a locally δ-sparse (n, d, r)-system. If d is large enough relative to δ, then

$$\gamma_\circ(\mathcal{S}) \leq \frac{1}{\delta}\left(\frac{n(\delta \log d)^{\frac{1}{r-1}}}{d^{\frac{1}{r-1}}} + \frac{2n}{d^{\frac{1}{r-1}}}\right).$$

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- However if I is maximal, then $X_j \subset I$ for all $j \neq i$, and therefore $|I| = (1 - 1/r)n$ for every maximal independent set I.
- Furthermore, $\mathcal{S}$ is an (n, d, r)-system with $d = (n/r)^{r-1}$.
- Note that $\mathcal{S}$ is not locally δ-sparse for any $\delta > 0$: in fact the number of chains of length two with ends $x, y \in X_1$ is roughly $d^{2-\frac{1}{r-1}}$.

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- If the plane has order q, then $\mathcal{S}$ is an (n, d, r)-system with $n = q^2 + q + 1$, $r = 3$ and $d = (q+1)\binom{q}{2}$.

Segre's problem II.

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- Best lower bound is roughly $2\sqrt{q}$, by Lunelli and Sce.
- Computational evidence by Fisher that the average size of a complete arc in $PG(2,q)$ is close to $\sqrt{3q\log q}$.
- Main open problem: finding lower bounds; in particular whether every complete arc has size at least $\sqrt{q}\omega(q)$ for some unbounded function $\omega(q)$.

Table of Contents

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- The focusing round at stage i involves selecting a subset $X(i)$ of atoms of the current set system $\mathcal{S}(i)$ with the property that there a very few chains of length $k \in \{1, 2\}$ joining any pair of vertices in the restriction of $\mathcal{S}(i)$ to $X(i)$.

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- Then in the construction round, we construct an independent set $\mathbb{I}(i)$ in $\mathcal{S}(i)$. This part is achieved via a randomized greedy algorithm, while keeping track of the atoms of $X'(i) := X \backslash X(i)$.

Outline of the proof II.

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- $\mathbb{I}$ is then a maximal independent set of size $\sum_{i=1}^{\mathbb{T}} |\mathbb{I}(i)| + O(\frac{n}{d}) \approx \frac{n(\log d)}{d} + O(\frac{n}{d})$.