Volume of hyperbolic octahedron with $\overline{3}$-symmetry

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Introduction

Calculating volumes of polyhedra is a classical problem, that has been well known since Euclid and remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is one of the main geometrical invariants for a 3-dimensional manifold.

Every 3-manifold can be presented by a fundamental polyhedron. That means we can pair-wise identify the faces of some polyhedron to obtain a 3-manifold. Thus the volume of 3-manifold is the volume of prescribed fundamental polyhedron.

It is known that regular hyperbolic octahedron with all vertices at infinity is a fundamental polyhedron for Whitehead link manifold. On the other hand, the minimal volume hyperbolic manifold and many others can be obtained by Dehn surgery along Whitehead link. Thus, hyperbolic octahedra can serve as fundamental polyhedra for a wide class of 3-manifolds, including hyperbolic manifolds of small volume. The latter seems especially interesting if we arrange hyperbolic manifolds in order of volume increasing.

Introduction

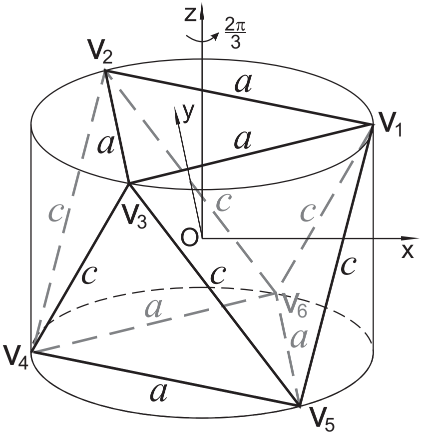
It is difficult problem to find the exact volume formulas for hyperbolic polyhedra of prescribed combinatorial type. It was done for hyperbolic tetrahedron of general type, but for general hyperbolic octahedron it is an open problem.

Nevertheless, if we know that a polyhedron has a symmetry, then the volume calculation is essentially simplified. Firstly this effect was shown by Lobachevskij. He found the volume of an ideal tetrahedron, which is symmetric by definition.

R.V. Galiulin, S.N. Mikhalev and I.Kh. Sabitov found the volumes of Euclidean octahedra with all possible types of symmetry, except the trivial one. The volumes of spherical octahedron with mmm or $2|m$-symmetry were given by N. Abrosimov, M. Godoy and A. Mednykh. The volume of hyperbolic octahedron with mmm-symmetry was obtained by N. Abrosimov and G. Baigonakova.

Definition

An octahedron has $\overline{3}$-symmetry if it admits the antipodal involution and order 3 rotation



$\overline{3}$ by Hermann Mauguin notation; S_6 by Schönflies notation;
$[2^+, 6^+]$ by Coxeter notation; $3\times$ by orbifold notation.

$$R = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since the symmetry group acts transitively on the set of vertices of octahedron $\mathcal{O}$, then we fix one vertex $v_1 = (r, 0, h)$ and consider its orbit under action of R and S:

$$\begin{array}{lll} v_1 = (r, 0, h), & v_2 = \left(-\frac{r}{2}, \frac{\sqrt{3}}{2}, h\right), & v_3 = \left(-\frac{r}{2}, -\frac{\sqrt{3}}{2}, h\right), \\ v_4 = (-r, 0, -h), & v_5 = \left(\frac{r}{2}, -\frac{\sqrt{3}}{2}, -h\right), & v_6 = \left(\frac{r}{2}, \frac{\sqrt{3}}{2}, -h\right). \end{array} \tag{1}$$

Now we know the coordinates of vertices of octahedron $\mathcal{O}$.

Using the coordinates of vertices, we compute edge lengths. Then we calculate normal vectors for faces and find the cosines of dihedral angles.

$$a^2 = 3\,r^2, \qquad \cos A = \frac{-r}{\sqrt{r^2+16\,h^2}} = \frac{-a}{\sqrt{3\,(4\,c^2-a^2)}},$$

$$c^2 = 4\,h^2 + r^2, \qquad \cos C = \frac{r^2-8\,h^2}{r^2+16\,h^2} = -\frac{a^2-2\,c^2}{a^2-4\,c^2}.$$

Remark

Let $\mathcal{O} = \mathcal{O}(a,c)$ is Euclidean octahedron with $\overline{3}$-symmetry. Then

(i) *edge lengths of $\mathcal{O}$ satisfy the condition* $0 < \dfrac{a^2}{c^2} < 3$;

(ii) *dihedral angles of $\mathcal{O}$ are related by equation* $2\sin\dfrac{C}{2} = \sqrt{3}\,\sin A$, *where* $A \in \left(\dfrac{\pi}{2}, \pi\right)$, $C \in \left(0, \dfrac{2\,\pi}{3}\right)$.

Converse is also true.

The volume of such an octahedron is $V = \dfrac{a^2}{3}\sqrt{3\,c^2-a^2}$

Caley-Klein model

Consider Minkowski space R^4_1 with scalar product
$\langle X, Y\rangle = -x_1\,y_1 - x_2\,y_2 - x_3\,y_3 + x_4\,y_4.$
The Caley-Klein model of hyperbolic space is the set of vectors
$K = \{(x_1, x_2, x_3, 1)\ :\ x_1^2 + x_2^2 + x_3^2 < 1\}$ forming the unit 3-ball in the hyperplane $x_4 = 1$.
The lines and planes in K are just the intersections of ball K with Euclidean lines and planes in the hyperplane $x_4 = 1$.
Let $V, W \in K$. Assume $V = (v, 1),\ W = (w, 1)$, where $v, w \in R^3$. Then the scalar product of V and W in Minkowski space is expressed via the Euclidean scalar product in R^3 by the formula $\langle V, W\rangle = 1 - \langle v, w\rangle_E$.
The distance between vectors V and W in Caley-Klein model is defined by equation

$$\operatorname{ch}\rho(V, W) = \frac{\langle V, W\rangle}{\sqrt{\langle V, V\rangle\,\langle W, W\rangle}}. \qquad (2)$$

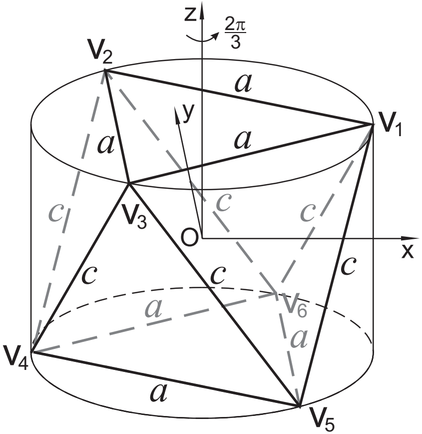
A plane in K is a set $\mathcal{P} = \{V \in K\ :\ \langle V, N\rangle = 0\}$, where $N = (n, 1)$, $\langle n, n\rangle_E > 0$ is a normal vector to the plane $\mathcal{P}$.

In model K consider two planes $\mathcal{P}, \mathcal{Q}$ with normal vectors N, M correspondingly. Then every of four dihedral angles between the planes $\mathcal{P}, \mathcal{Q}$ is defined by relation

$$\cos\widehat{(\mathcal{P}, \mathcal{Q})} = \pm \frac{\langle N, M \rangle}{\sqrt{\langle N, N \rangle \langle M, M \rangle}}. \tag{3}$$

Now let $V_1 = (v_1, 1)$, $V_2 = (v_2, 1)$, $V_3 = (v_3, 1)$ are three non-coplanar vectors in K. Then there is a unique plane passes through them: $\mathcal{P} = \{V \in K : \langle V, N \rangle = 0\}$ with normal vector $N = (n, 1)$, where the coordinates of vector $n \in R^3$ are uniquely determined as the solution of a system of linear equations

$$\begin{aligned} \langle v_1, n \rangle_E - 1 &= 0, \\ \langle v_2, n \rangle_E - 1 &= 0, \\ \langle v_3, n \rangle_E - 1 &= 0. \end{aligned}$$



$$V_i = (v_i, 1), \quad \overline{R} = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overline{S} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In Caley-Klein model K consider vectors $V_i = (v_i, 1)$, $i = 1, \dots, 6$, where v_i are the vertices of Euclidean octahedron with $\overline{3}$-symmetry, studied before. The isometries R, S of Euclidean 3-space are naturally extended to isometries of hyperbolic space K:

$$\overline{R} : (v, 1) \to (R\, v, 1), \quad \overline{S} : (v, 1) \to (S\, v, 1).$$

As before, $\overline{R}$ is an order 3 rotation around the axe Ox_3, and $\overline{S}$ is an antipodal involution at point (0,0,0,1), which is the centre of K. In contrast to Euclidean case, we have one additional condition $r^2 + h^2 < 1$. Or equivalently, all the vertices $V_i \in K$. Without lost of generality, assume that $r > 0$, $h > 0$. Then the vectors $V_i = (v_i, 1)$, $i = 1, \dots, 6$, are determine the vertices of a hyperbolic octahedron $\mathcal{O}$ with $\overline{3}$-symmetry. Using the coordinates of vertices we compute the edge lengths by (2):

$$\operatorname{ch} a = \frac{2 + r^2 - 2\, h^2}{2\,(1 - r^2 - h^2)}, \quad \operatorname{ch} c = \frac{2 - r^2 + 2\, h^2}{2\,(1 - r^2 - h^2)}. \tag{4}$$

Solving (7) with respect to r^2, h^2 we get

$$r^2 = \frac{4\,(\operatorname{ch} a - 1)}{3\,(\operatorname{ch} a + \operatorname{ch} c)}, \quad h^2 = \frac{3 \operatorname{ch} c - \operatorname{ch} a - 2}{3\,(\operatorname{ch} a + \operatorname{ch} c)}. \tag{5}$$

Proposition

A hyperbolic octahedron $\mathcal{O} = \mathcal{O}(a, c)$, admitting $\overline{3}$-symmetry, with edge lengths a, c is exist if and only if

$$3 \operatorname{ch} c - \operatorname{ch} a - 2 > 0 .$$

Using coordinates of vertices V_i, $i = 1, \ldots, 6$, by formula (3) we get the cosines of dihedral angles up to choice of signs. But in particular case of regular hyperbolic octahedron we already know the answer. This allows us to choose the signs correctly.

Proposition

$$\cos A = \frac{(\operatorname{ch} c - \operatorname{ch} a - 1)\sqrt{\operatorname{ch} a - 1}}{\sqrt{(1 + 2\operatorname{ch} a)(2\operatorname{ch}^2 c - \operatorname{ch} a - 1)}},$$

$$\cos C = \frac{1 - \operatorname{ch} c + \operatorname{ch} a \operatorname{ch} c - \operatorname{ch}^2 c}{2\operatorname{ch}^2 c - \operatorname{ch} a - 1}. \qquad (6)$$

We differentiate the volume as a composite function

$$\begin{aligned}\frac{\partial V}{\partial a} &= \frac{\partial V}{\partial A}\,\frac{\partial A}{\partial a} + \frac{\partial V}{\partial C}\,\frac{\partial C}{\partial a},\\ \frac{\partial V}{\partial c} &= \frac{\partial V}{\partial A}\,\frac{\partial A}{\partial c} + \frac{\partial V}{\partial C}\,\frac{\partial C}{\partial c}.\end{aligned} \tag{7}$$

By Schläfli formula we have

$$dV = -\sum_{\theta} \frac{\ell_\theta}{2}\, d\theta = -3\,a\,dA - 3\,c\,dC,$$

hence $\dfrac{\partial V}{\partial A} = -3\,a, \quad \dfrac{\partial V}{\partial C} = -3\,c.$

We substitute all the expressions into formulas (7). Finally we get

$$
\begin{aligned}
f(a,c) &:= \frac{\partial V}{\partial a} = \frac{3\,(a\,F + (1+2\operatorname{ch} a)\,c\,G)}{(1+2\operatorname{ch} a)\Delta},\\
g(a,c) &:= \frac{\partial V}{\partial c} = \frac{3\,(a\,G + c\,H)}{\Delta}.
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
F &= 1 + 2\operatorname{ch} a + 2\operatorname{ch}^2 a + \operatorname{ch}^3 a - 2\operatorname{ch} c - 2\operatorname{ch} a\operatorname{ch} c - \operatorname{ch}^2 c\\
&\quad - \operatorname{ch} a\operatorname{ch}^2 c + \operatorname{ch}^2 a\operatorname{ch} c - 4\operatorname{ch}^2 a\operatorname{ch}^2 c + 3\operatorname{ch}^3 c\,,\\
G &= \operatorname{sh} a\operatorname{sh} c\,(-1 + 2\operatorname{ch} a),\\
H &= (1 - \operatorname{ch} a)(1 + \operatorname{ch} a + 2\operatorname{ch} c\,(-1 + \operatorname{ch} c)),\\
\Delta &= (\operatorname{ch} 2c - \operatorname{ch} a)\sqrt{(-2 - \operatorname{ch} a + 3\operatorname{ch} c)(\operatorname{ch} a + \operatorname{ch} c)}.
\end{aligned}
$$

Remark

In the domain of existence of the octahedron $\mathcal{O}$ with $\overline{3}$-symmetry

$$\Omega = \{(\operatorname{ch} a, \operatorname{ch} c) : \operatorname{ch} a > 1,\ \operatorname{ch} c > 1,\ 3\operatorname{ch} c - \operatorname{ch} a - 2 > 0\}$$

the quantities G and Δ are positive, H is negative, and F changes the sign.

The boundary of Ω consists of two rays $\{\operatorname{ch} a = 1, \operatorname{ch} c \geq 1\}$ and $\{\operatorname{ch} a \geq 1, 3 \operatorname{ch} c = \operatorname{ch} a + 2\}$. The octahedron degenerate on each of these rays into the line segment or planar hexagon. Hence, $V = 0$ on the boundary of Ω.

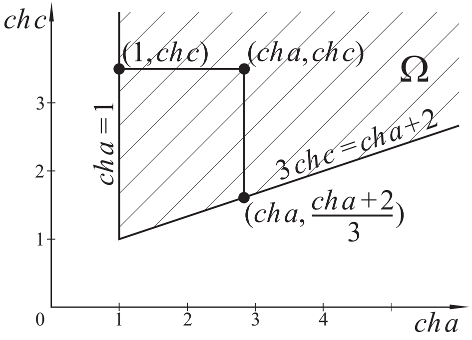


Рис.: Domain of existence Ω of octahedron $\mathcal{O} = \mathcal{O}(a, c)$

Now we integrate the differential form $dV = f(a, c)\, da + g(a, c)\, dc$ along horizontal or vertical segment from the boundary of Ω to the point $(\operatorname{ch} a, \operatorname{ch} c)$. Thus we obtain the main theorem.

Theorem

The volume of hyperbolic octahedron $\mathcal{O} = \mathcal{O}(a, c)$ *with* $\overline{3}$*-symmetry is given by each of the following two formulas*

(i) $$V = \int_0^a f(a, c)\, da\,,$$

(ii) $$V = \int_{arch(\frac{\operatorname{ch} a+2}{3})}^c g(a, c)\, dc\,,$$

where

$$f(a, c) = \frac{3\,(a\,F + (1 + 2\operatorname{ch} a)\,c\,G)}{(1 + 2\operatorname{ch} a)\Delta}, \quad g(a, c) = \frac{3\,(a\,G + c\,H)}{\Delta},$$

$$F = 1 + 2\operatorname{ch} a + 2\operatorname{ch}^2 a + \operatorname{ch}^3 a - 2\operatorname{ch} c - 2\operatorname{ch} a\operatorname{ch} c - \operatorname{ch}^2 c$$
$$- \operatorname{ch} a\operatorname{ch}^2 c + \operatorname{ch}^2 a\operatorname{ch} c - 4\operatorname{ch}^2 a\operatorname{ch}^2 c + 3\operatorname{ch}^3 c\,,$$

$$G = \operatorname{sh} a\operatorname{sh} c\,(-1 + 2\operatorname{ch} a),$$

$$H = (1 - \operatorname{ch} a)(1 + \operatorname{ch} a + 2\operatorname{ch} c\,(-1 + \operatorname{ch} c)),$$

$$\Delta = (\operatorname{ch} 2c - \operatorname{ch} a)\sqrt{(-2 - \operatorname{ch} a + 3\operatorname{ch} c)(\operatorname{ch} a + \operatorname{ch} c)}.$$

Thank you for attention!