Arc-types of Vertex-Transitive Graphs

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Graph symmetries

An automorphism of a (simple) graph $X = (V,E)$ is any bijection $\theta: V \rightarrow V$ on the vertex set $V$, preserving adjacency (so preserving the edge set $E$).

A crude measure of the ‘symmetry’ of the graph $X$ is the order of its automorphism group $A = \text{Aut}(X)$.

The graph is called vertex-transitive if $A = \text{Aut}(X)$ has a single orbit on $V = V(X)$, or edge-transitive if $A$ has a single orbit on $E = E(X)$, or arc-transitive (or symmetric) if $A$ has a single orbit on the set of arcs — where an arc is an ordered pair $(v,w)$ of adjacent vertices.
Special cases of VT graphs

Let $X = (V, E)$ be a vertex-transitive graph, and let $A_v$ be the stabiliser in $A = \text{Aut}(X)$ of a vertex $v$ of $X$, and let $X(v)$ be the neighbourhood of $v$. Then $X$ is:

- **arc-transitive** (or symmetric) if $A_v$ is transitive on $X(v)$
- **zero-symmetric** (or a GRR of $A$) if $A_v$ is trivial
- **$\frac{1}{2}$-arc-transitive** if $X$ is edge-transitive but not arc-transitive.

GRR $\equiv$ ‘Graphical regular representation’
  $\equiv$ Cayley graph with no extra automorphisms.

In the third case, if $w$ is any neighbour of $v$, then the two arcs $(v, w)$ and $(w, v)$ associated with the edge $\{v, w\}$ lie in different orbits of $A$, so $A_v$ has two ‘paired’ orbits on $X(v)$. 
‘Pairing’ and ‘self-pairing’

Let $P$ be a transitive permutation group on a set $\Omega$, and let $P_v = \{ g \in P \mid v^g = v \}$, the stabiliser in $P$ of the point $v$.

An orbit $\Delta$ of $P$ on $\Omega \times \Omega$ is called an orbital of $P$, and for any such orbital $\Delta$, the set $\Delta(v) = \{ w \in \Omega \mid (v, w) \in \Delta \}$ is an orbit of $P_v$ (which is also called a sub-orbit of $P$ for $v$). If $(v, w) \in \Delta$ then the orbital $\Delta'$ containing $(w, v)$ is called the paired orbital of $\Delta$, and if $\Delta' = \Delta$ then $\Delta$ is self-paired.

In the VT graph context, we consider paired sub-orbits of $A = \text{Aut}(X)$ with respect to a vertex $v$. The orbits of $A_v$ containing the arcs $(v, w)$ and $(v, w')$ are paired if $(v, w')$ is in the same orbit of $A$ on arcs as $(w, v)$, and the orbit of $(v, w)$ is self-paired if $(v, w)$ is reversed by some automorphism of $X$. 
Example

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- \( \Delta_1 \) (length 3), self-paired
- \( \Delta_2 \) (length 2), paired with \( \Delta_4 \)
- \( \Delta_3 \) (length 2), self-paired
- \( \Delta_4 \) (length 2), paired with \( \Delta_2 \)
- \( \Delta_5 \) (length 1), self-paired
The arc-type of a VT graph

Let $X = (V, E)$ be a vertex-transitive graph of valency $d$.

The arc-type of $X$ is defined in terms of the orbits of the vertex-stabiliser $A_v$ on the neighbourhood $X(v)$.

Specifically, the arc-type is the partition of $d$ as the sum

$$n_1 + \cdots + n_t + (m_1 + m_1) + \cdots + (m_s + m_s)$$

where the $n_i$ are the lengths of the self-paired orbits of $A_v$, and the $m_j$ are the lengths of the non-self-paired orbits, in descending order.
Important cases

• If $X$ is arc-transitive, then its arc-type is $d$ (since $A_v$ has a single orbit on $X(v)$)

• If $X$ is a GRR, then all $m_i = 1$ and all $n_j = 1$ (since $A_v$ is trivial)

• If $X$ is half-arc-transitive, then its arc-type is $(d/2 + d/2)$ (since $A_v$ has a dual pair of orbits on $X(v)$).
Small valency cases

- **Valency 1**: $X \cong K_2$ (or a ladder graph), with arc-type 1

- **Valency 2**: $X$ is a cycle (or union of cycles), so is AT, with arc-type 2

- **Valency 3**: the arc-type of $X$ is one of the following:
  - $3 \ldots$ e.g. $K_4$ (arc-transitive)
  - $2+1 \ldots$ e.g. triangular prism
  - $1+1+1 \ldots$ e.g. Cayley graph for a group of order 18 generated by three involutions
  - $1+(1+1) \ldots$ e.g. Cayley graph for a group of order 20 generated by an involution and a non-involution.
Arc-type $1 + 1 + 1$  

Arc-type $1 + (1 + 1)$
• **Valency 4:** the arc-type of $X$ is one of the following:

  4 ... e.g. $K_5$ (arc-transitive)
  3 + 1 ... e.g. Cartesian product $K_4 \Box K_2$ (order 8)
  (2 + 2) ... e.g. Holt graph (half-arc-transitive)
  2 + 2 ... e.g. a 4-valent Cayley graph for $\mathbb{Z}_7$
  2 + (1 + 1) ... e.g. a 4-valent graph of order 40
  2 + 1 + 1 ... e.g. a variant of the hexagonal prism
  (1 + 1) + (1 + 1) ... e.g. a GRR for $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$
  1 + 1 + (1 + 1) ... e.g. a GRR for $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
  1 + 1 + 1 + 1 ... e.g. a GRR for $D_8$ (order 16).

**Obvious question:** What arc-types are realisable?
**Theorem** [MC, TP & AŽ (Nov/Dec 2014)]

Every arc-type other than $1 + 1$ and $(1 + 1)$ is realisable by some VT graph.

The proof uses several families of examples of VT graphs, and a construction showing that the arc-type of a Cartesian product of two relatively prime graphs is the natural sum of their arc-types.
**Cartesian products**

The Cartesian product $X \boxtimes Y$ of graphs $X$ and $Y$ is a graph with vertex set $V(X) \times V(Y)$, and vertices $(u, x)$ and $(v, y)$ adjacent iff $u = v$ and $x \sim y$ in $Y$, or $x = y$ and $u \sim v$ in $X$.

**Properties:**

- $X \boxtimes Y$ is **connected** if and only if $X$ and $Y$ are connected.
- If $X$ and $Y$ are **regular** with valencies $d$ and $e$, then $X \boxtimes Y$ is regular with valency $d + e$.
- $\text{Aut}(X \boxtimes Y)$ has a subgroup isomorphic to $\text{Aut}(X) \times \text{Aut}(Y)$.
- $X \boxtimes Y$ is **VT** if and only if $X$ and $Y$ are **VT**.

Similar properties hold for larger products $X_1 \boxtimes X_2 \ldots \boxtimes X_m$. 
Prime and relatively prime graphs

A graph $X$ is prime (w.r.t. Cartesian product) if it is not isomorphic to a Cartesian product of two smaller graphs.

Every connected finite graph can be decomposed as a Cartesian product of prime graphs, uniquely up to the order of the factors – see Imrich & Klavzar (2000).

Two graphs are relatively prime (w.r.t. Cartesian product) if they have no non-trivial common factor.

**Theorem** [Imrich & Klavzar (2000)] If $X$ is the Cartesian product $X_1 \square \ldots \square X_k$ of connected relatively prime graphs, then $\text{Aut}(X) \cong \text{Aut}(X_1) \times \cdots \times \text{Aut}(X_k)$. 
Consequence for arc-types:

**Theorem [TP?]**

Let $X_1, \ldots, X_k$ be non-trivial connected finite graphs that are vertex-transitive, with arc-types $\tau_1, \ldots, \tau_k$, respectively. If $X_1, \ldots, X_k$ are relatively prime, then also $X = X_1 \Box \ldots \Box X_k$ is vertex-transitive, and the arc-type of $X$ is $\tau_1 + \cdots + \tau_k$.

This gives a natural means of constructing VT graphs with prescribed arc-type, from smaller examples.

Also helpful is a test for checking whether a given graph $X$ is a Cartesian product of two smaller graphs. A necessary condition is that every edge of $X$ lies in a 4-cycle. Similarly, the edges of any 3-cycle must all lie in the same factor.
Constructing graphs with given arc-type

To construct VT graphs with (almost) arbitrary arc-type, we can use the additivity of arc-types for Cartesian products of relatively prime graphs.

For example, arc-type $3 + 2 + 1$ can be realised by taking the Cartesian product of two relatively prime VT cubic graphs with arc-types 3 and $2 + 1$.

Similarly, the valency 13 arc-type $1 + 1 + 1 + \ldots + 1$ can be realised by taking the Cartesian product of four relatively prime VT graphs with arc-types $1 + 1 + 1 + 1$, $1 + 1 + 1$, $1 + 1 + 1$ and $1 + 1 + 1$.

This approach requires a good supply of ‘building blocks’.
Some interesting building blocks

1) For every valency $d \geq 2$, there exist infinitely many prime graphs of arc-type $d$.

Cycle graphs $C_n$ for $n \geq 5$ are prime graphs of arc-type 2.

For $d \geq 3$, by a theorem of Macbeath there are infinitely many primes $p$ for which $G = \text{PSL}(2, p)$ is generated by two elements $x$ and $y$ such that $x$, $y$ and $xy$ have orders 2, $d$ and $d + 4$, respectively. Then for each $p$, the double coset graph $\Gamma(G, \langle y \rangle, x)$ is a prime graph, with arc-type $d$.

[Other examples can be constructed as covers of small ones.]
Building blocks (cont.)

2) For every integer $d \geq 2$, there exist infinitely many prime graphs of arc-type $(d + d)$.

In 1970 Bouwer constructed a family of vertex- and edge-transitive graphs $B(k, m, n)$, using modular arithmetic. These graphs have order $mn^{k-1}$ and valency $2k$, for any $k \geq 2$.

Bouwer also proved that the graphs $B(k, 6, 9)$ are half-arc-transitive for all $k$. His approach was modified (by MC & AŽ) to show that almost all $B(k, m, n)$ are half-arc-transitive.

In particular, if $m > 6$ and $n > 7$ (and $2^m \equiv 1 \mod n$), the Bouwer graph $B(k, m, n)$ is half-arc-transitive of girth 6 for all $k$. This gives infinitely many prime graphs of valency $2k$. 
Building blocks (cont.)

3) For every integer \( m \geq 2 \), there exist infinitely many prime graphs of arc-type \( m + 1 \).

Take a cycle \( C_{2n} \) of even length, with vertices 0, 1, ..., \( 2n - 1 \), and construct a thickened \( m \)-cover of it, by replacing each vertex by \( m \) vertices, and each edge \{\( i-1, i \} \) by \( mK_2 \) (that is, \( m \) parallel edges) if \( i \) is odd, or by \( K_{m,m} \) (complete bipartite) if \( i \) is even. The resulting graph is a prime VT graph with arc-type \( m + 1 \).
Building blocks (cont.)

4) For every integer $m \geq 2$ there exists a prime graph of arc-type $m + (1 + 1)$.

Take a particular thickened $m$-cover of the VT graph of arc-type $1 + (1 + 1)$ considered earlier.

5) For every integer $m \geq 2$ there exists a prime graph of arc-type $(m + m) + 1$.

Take a different thickened $m$-cover of the VT graph of arc-type $1 + (1 + 1)$ considered earlier.
Building blocks (cont.)

6) Infinitely many prime graphs of arc-type \((1+1)+(1+1)\).

For any prime \(p \equiv 1 \mod 6\), take the Cayley graph for the group \(G = \mathbb{Z}_p \rtimes_k \mathbb{Z}_6 = \langle a, b \mid a^6 = b^p = 1, a^{-1}ba = b^k \rangle\), where \(k\) is a primitive 6th root of 1 mod \(p\), with generators \(x = a\) and \(y = ba^2\). This is a GRR, with the orbits corresponding to \(x\) and \(x^{-1}\) being paired, as for those for \(y\) and \(y^{-1}\).

7) For every integer \(m \geq 2\) there exists a prime graph of arc-type \((m + m) + (1 + 1)\).

Take a particular thickened \(m\)-cover of the VT graph of arc-type \(1 + 1 + (1 + 1)\) considered above with \(p = 7\).
Building blocks (cont.)

8) Infinitely many prime graphs of arc-type $1 + 1 + 1$

For each odd integer $n = 2m - 1 \geq 11$, take the Cayley graph for the symmetric group $S_n$ with three involutory generators:

$$x_1 = (1, 2)(3, n)(4, n - 1) \ldots (m, m + 2),$$
$$x_2 = (2, n)(3, n - 1)(4, n - 2) \ldots (m, m + 1),$$
$$x_3 = (1, n - 1)(2, n - 2)(3, n - 3) \ldots (m - 1, m),$$

carefully chosen so as to make the Cayley graph a GRR with three edge orbits.
9) Prime graphs of miscellaneous arc-types:

- $1+1+(1+1)$ ... e.g. Cayley graph for $\mathbb{Z}_5 \times \mathbb{Z}_4$ (order 20)
- $1+1+1+1$ ... e.g. Cayley graph for $D_8$ (of order 16)
- $1+(1+1)$ ... e.g. GRRs by Coxeter, Frucht & Powers
- $(1+1)+(1+1)+(1+1)$ ... e.g. Cayley graph for $\text{SL}(2, 3)$ of order 24, found with the help of MAGMA.
Example of arc-type \((1 + 1) + (1 + 1) + (1 + 1)\)
Final comments

Taking Cartesian products of these building blocks gives a proof that every arc-type is possible except 1+1 and (1+1).

We can also define the edge-type of a $d$-valent VT graph $X$ to be the partition $d = k_1 + \cdots + k_r$, where $k_i = 2|\Delta_i|/|V(X)|$ is the size of the restriction of the $i$th orbit $\Delta_i$ on $E(X)$ to the set of edges incident with any $v \in V(X)$.

Corollary: Every edge type is possible except 1+1.
THANK YOU