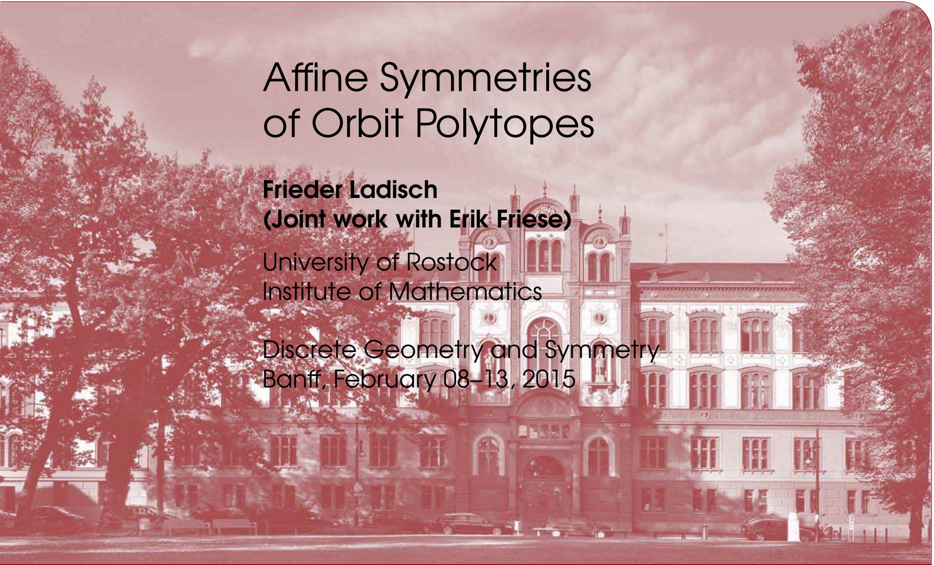

Affine Symmetries of Orbit Polytopes

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University of Rostock
Institute of Mathematics

Discrete Geometry and Symmetry
Banff, February 08–13, 2015

Definitions

Definition

Let $G \leq \mathrm{GL}(d, \mathbb{R})$ be a finite group and $v \in \mathbb{R}^d$.
The orbit polytope $P(G, v)$ of G at v is the convex hull

$$P(G, v) := \mathrm{conv}\{gv \mid g \in G\}.$$

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Definition

An affine symmetry of a polytope $P \subset \mathbb{R}^d$ is a bijection $P \to P$ which is the restriction of an affine map (of $\mathbb{R}^d$, say).

General Observations

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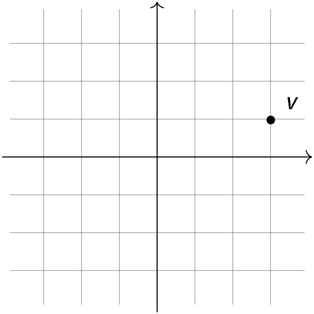
- Each $g \in G$ yields an affine symmetry of $P(G, v)$.
- Depending on G and on v, there may be more affine symmetries or not.

Example: Dihedral group D_4, I. (“Generic” points)

$$G = \left\langle t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_4,\ |G| = 8.$$



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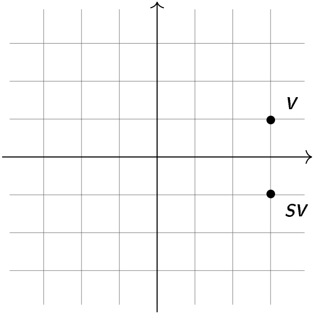

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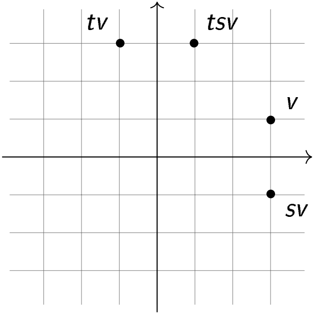

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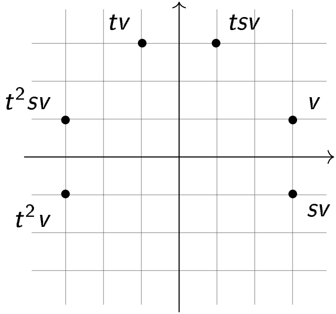

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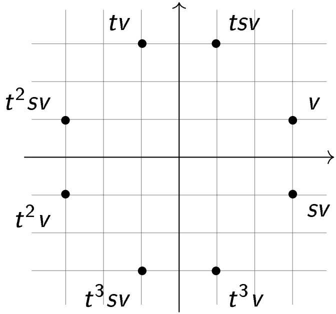

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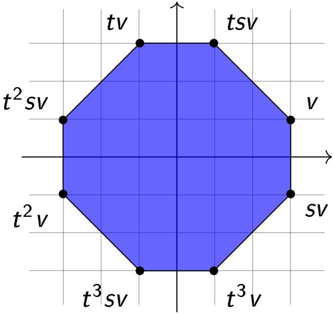

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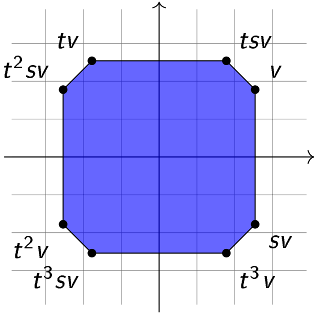

(Another v)

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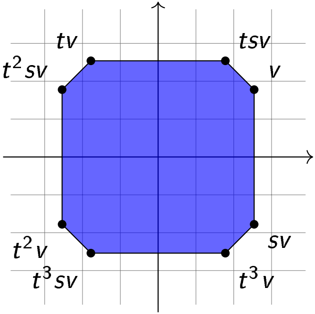

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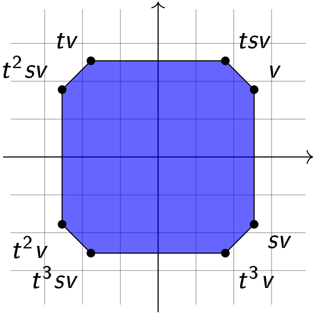

- $\mathrm{AGL}(P(G, v)) = G$
- "typical" (generic) orbit polytope.

Example: Dihedral group D_4, II. (Special points)

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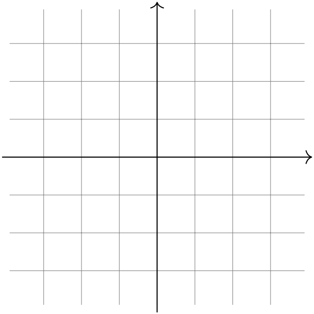

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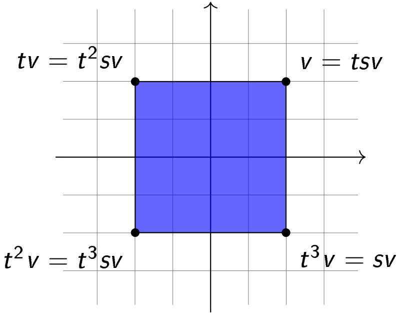

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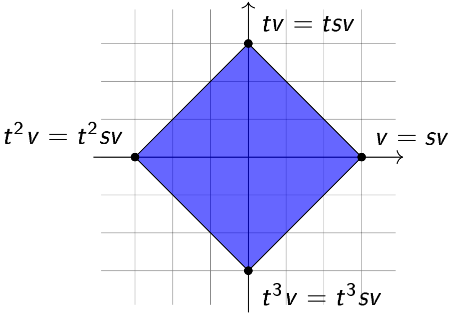

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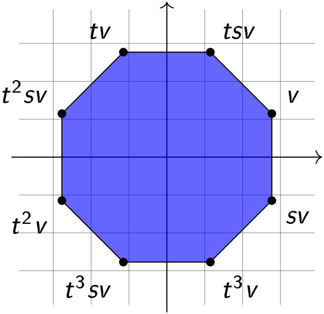

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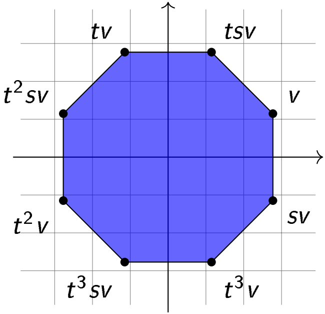

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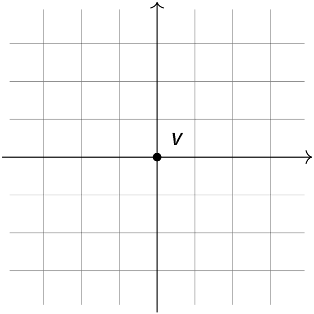

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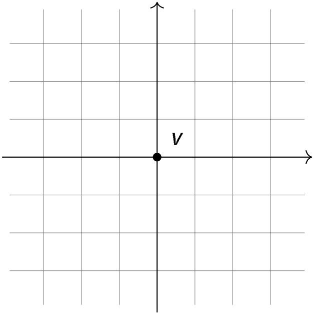

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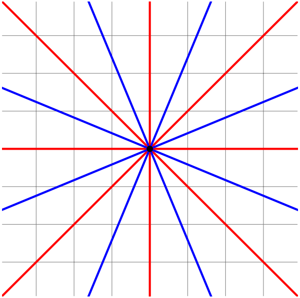
(Here only for $v = (0,0)$.)

Example: Dihedral group D_4, III. (Summary)



Different types of orbit polytopes:

I. Fewer vertices. (red)
II. More symmetries. (blue)
III. Smaller dimension.
IV. “Generic”. (everything else)

Non-generic Points

Assumption: There is at least one v such that $P(G, v)$ is full-dimensional.

- The set of v's such that $P(G, v)$ is not full-dimensional is the zero-set of some polynomials.
- The set of v's having non-trivial stabilizer in G is a finite union of proper subspaces.
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First aim: explain last point and precisely define "generic".

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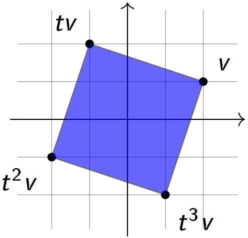
- If $|Gv| = |G|$, we may identify the symmetry with a permutation of G.
- Conversely, suppose permutation π of G is given:
- For which $v \in \mathbb{R}^d$ is there an affine map sending gv to $\pi(g)v$?
- For which π is this true for all full-dimensional orbit polytopes $P(G, v)$?
  Example $G = D_4$: only for perms of form $g \mapsto hg$, $h \in G$.

Example: Group generated by a rotation

$$G = \left\langle t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong C_4$$



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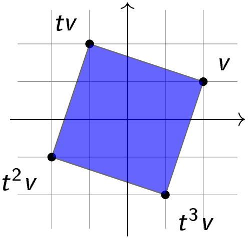

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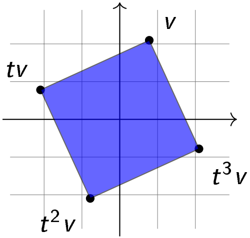

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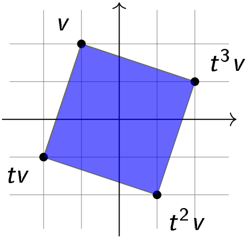

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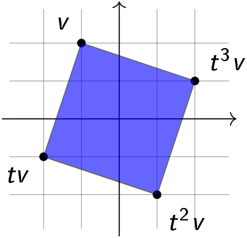

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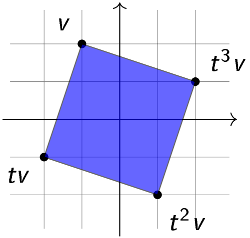

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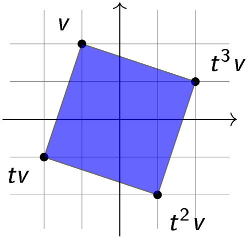
- $\implies$ "generic" symmetry (t, t^3)
- "Implication phenomenon": C_4 implies more symmetries.



Generic symmetries

- The set of permutations π of G such that $gv \mapsto \pi(g)v$ is affine for all v with $P(G, v)$ full-dimensional is called the generic symmetry group of G.
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Theorem

The set of $v \in \mathbb{R}^d$ such that

$$P(G, v) \text{ full-dimensional} \quad \text{and} \quad |\mathrm{AGL}(P(G, v))| > |\mathrm{GenSym}(G)|$$

is contained in the zero set of some (non-zero) polynomials.
$\mathrm{AGL}(P(G, v)) \cong \mathrm{GenSym}(G)$ *for "almost all" v with $P(G, v)$ full-dimensional.*

Generic points

Definition

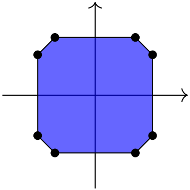
A point $v \in \mathbb{R}^d$ is called generic (for G),
and $P(G, v)$ is called a generic orbit polytope, if

- $P(G, v)$ is full-dimensional,
- the stabilizer of v in G is trivial,
- $|\mathrm{AGL}(P(G, v))| = |\mathrm{GenSym}(G)|$.

Last theorem says: The non-generic points form a proper algebraic subset.

Examples so far

- $G = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong D_4$
- Generic symmetry group: just G.



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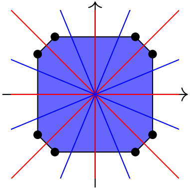

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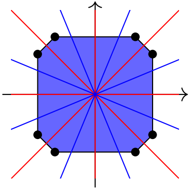

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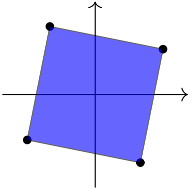

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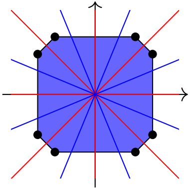

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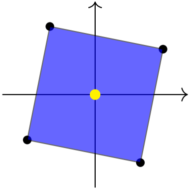

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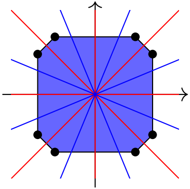

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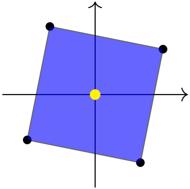
- not generic: only the origin
- $\mathrm{AGL}(P(G, v))$ depends on v, but not its isomorphism type.



Another example

- $G = \left\langle \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix} \right\rangle \cong D_4.$
- $P(G, v)$ not full-dimensional if $v = (v_1, v_2, v_3, v_4)^t$ with $\begin{vmatrix} v_1 & v_3 \\ v_2 & v_4 \end{vmatrix} = 0.$
- All other points are generic.
- Generic symmetry group of order 384.
- In fact, all generic orbit polytopes are affinely equivalent to the 4-dimensional cross polytope.

Symmetry groups of generic orbit polytopes

Theorem

Let $P(G, w)$ be a full-dimensional orbit polytope for G.
Then the affine symmetry group of $P(G, w)$ contains a conjugate of the affine symmetry group of each generic orbit polytope:

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If v generic, then

$$S^{-1}\,\mathrm{AGL}(P(G,v))\,S \leq \mathrm{AGL}(P(G,w)) \quad \textit{for some} \quad S \in \mathrm{GL}(d,\mathbb{R}).$$

In particular, generic orbit polytopes have similar affine symmetry groups.

Different generic orbit polytopes

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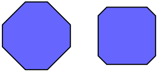
- In general: different generic orbit polytopes are not affinely equivalent.

  (Example: D_4 in its natural representation.)



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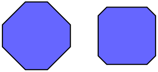
- Different generic orbit polytopes have similar affine symmetry group.
- In general: different generic orbit polytopes are not affinely equivalent.

  (Example: D_4 in its natural representation.)



- Even worse: different generic orbit polytopes may have different face lattices! (They are not combinatorially equivalent.)
  - Example by Onn (1993): S_4 acting on $\mathbb{R}^5$, generic orbit polytopes with different face lattices.

Generic closure

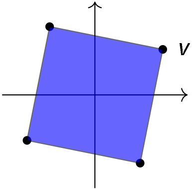
A corollary of the last theorem is:

Corollary

The generic symmetry group "grows at most once".

Explanation by example:

- $G = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong C_4$.
- $\widehat{G} := \mathrm{AGL}(P(G, v)) \cong D_4$.
- v is not generic for $\widehat{G}$.
- But: $\mathrm{AGL}(P(\widehat{G}, w)) = \widehat{G}$,
  if w generic for $\widehat{G}$.



Generically closed

Question

Which groups G have a larger generic symmetry group?

Such groups G imply additional symmetries.

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Such groups G imply additional symmetries.
If $\mathrm{AGL}(P(G, v)) = G$, then G is called generically closed.
Outcome: G generically closed in "most cases".
e.g., in dimensions 2 and 3: *only* the abelian groups are not generically closed.

Abelian groups

Theorem

Let $G \leq \mathrm{GL}(d, \mathbb{R})$ be abelian. Then all full-dimensional orbit polytopes are generic and affinely equivalent to each other.

Abelian groups

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Theorem

Let $G \leq \mathrm{GL}(d, \mathbb{R})$ be abelian.
A permutation $\pi : G \to G$ is a generic symmetry if and only if

$$\mathrm{tr}\left(\pi(g)^{-1}\pi(h)\right) = \mathrm{tr}(g^{-1}h) \quad \textit{for all} \quad g, h \in G.$$

Abelian groups (continued)

Corollary

Let G be abelian.

- *The permutation $g \mapsto g^{-1}$ is in $\mathrm{GenSym}(G)$.*
- *If $g^2 \neq 1$ for some $g \in G$, then G is not generically closed.*

(There are elementary abelian 2-groups of sizes $2^5, 2^6, \ldots$ which are generically closed.)

Abelian groups (continued)

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Remark: The results for abelian groups carry over (with modifications) to a larger class of groups, where the polytopes are representation polytopes.

Computing the generic symmetry group in general

Theorem

Given a group $G \leq \mathrm{GL}(d, \mathbb{R})$, we know how to compute the generic symmetry group from the character of G, that is, the function $G \ni g \mapsto \mathrm{tr}(g)$.
(The exact statement is too technical to give here.)

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With this result, one can answer the following question for any given finite abstract group G:

Question

For which representations $\rho\colon G \to \mathrm{GL}(d, \mathbb{R})$ *is the image generically closed?*

A question of Babai

Question

Which abstract finite groups G are isomorphic to a generically closed finite matrix group?

The last question is equivalent to a question of Babai:

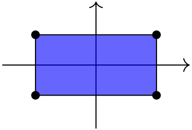
Question (Babai 1977)

Which abstract finite groups are isomorphic to the affine symmetry group of an orbit polytope?

- Babai (1977) determined the finite groups which are isomorphic to the orthogonal symmetry group of an orbit polytope.

Orthogonal and affine symmetries

- If a group is isomorphic to the affine symmetry group of an orbit polytope, then it is isomorphic to the orthogonal symmetry group of an orbit polytope.
- The converse is wrong.
  - Example: $V_4 \cong \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$.
  - orthogonal symmetry group V_4, affine D_4.
  - other possible orbit polytope of V_4: 3-simplex.



- We only know the following examples that the converse is wrong: $V_4 = (C_2)^2$, $(C_2)^3$ and $(C_2)^4$.
- Are there any others?

Combinatorial equivalences

Question

For which groups G are all generic orbit polytopes combinatorially equivalent?

True in the following cases:

- G abelian. (The generic orbit polytopes are even affinely equivalent.)
- More generally, in the representation polytope situation.
- G is a finite reflection group.
  (The orbit polytopes are the so-called permutahedrons.)
- The dimension is at most 4.

Combinatorial symmetries

Recall: A combinatorial symmetry of a polytope is a permutation of the vertices respecting the face lattice.

Question

Let $P(G, v)$ be a full-dimensional orbit polytope. Is there always a point v_0 such that $P(G, v)$ and $P(G, v_0)$ are combinatorially equivalent and such that all combinatorial symmetries of $P(G, v_0)$ are affine symmetries of $P(G, v_0)$?

True if $P(G, v)$ is combinatorially regular (McMullen 1967).

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True if $P(G,v)$ is combinatorially regular (McMullen 1967).
A consequence would be a “Yes” to:

Question

Is every combinatorial symmetry of an representation polytope an affine symmetry?

Paper: Erik Friese and Frieder Ladisch,
Affine symmetries of orbit polytopes,
arxiv:1411.0899 [math.MG] (2014)

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Thank you for your attention!

Example: rotation group of the tetrahedron

$$G = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong A_4$$

point	polytope	symmetry group
$(1,1,1)$	regular tetrahedron	S_4
$(1,0,0)$	regular octahedron	$S_4 \times C_2$
$(1,1,2)$	truncated tetrahedron	S_4
$(1,1,0)$	cuboctahedron	$S_4 \times C_2$
$(0,1,3)$	skew icosahedron	$A_4 \times C_2$
$(0,1,\varphi)$	regular icosahedron	$A_5 \times C_2$
$(0,0,0)$	point	id
generic	skew icosahedron	A_4

Representation polytopes

- A representation polytope $P(G)$ is the convex hull of a finite matrix group $G \leq \mathrm{GL}(d, \mathbb{R})$.
- Example: Birkhoff polytope (generated by all permutation matrices).
- Representation polytopes are orbit polytopes: $P(G) = P(G, I)$.

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- Example: Birkhoff polytope (generated by all permutation matrices).
- Representation polytopes are orbit polytopes: $P(G) = P(G, I)$.

Theorem

- *I is generic in the space generated by G (for G acting by left multiplication)*
- *All full-dimensional orbit polytopes in this space are generic, and are affinely equivalent to each other.*

(If G is an abelian group, then every orbit polytope is affinely equivalent to a representation polytope.)

Affine symmetries of representation polytopes

- Let $G \leq \mathrm{GL}(d, \mathbb{R})$ be a finite matrix group.
- Obvious affine symmetries of $P(G)$: $g \mapsto hg$, $g \mapsto gh$, $g \mapsto g^{-1}$.

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- Let γ be the character of G acting on the linear hull of G in $\mathbb{R}^{d \times d}$.
- (γ has form $\gamma = \sum_{\chi} \chi(1)\chi$, the sum running over some subset of $\mathrm{Irr}(G)$.)

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Theorem

A permutation $\pi : G \to G$ is induced by an affine symmetry of the representation polytope $P(G)$ if and only if

$$\gamma\left(\pi(g)^{-1}\pi(h)\right) = \gamma(g^{-1}h) \quad \text{for all} \quad g, h \in G.$$

Computing generic symmetries from the character

Technical preliminaries

- Given: finite group $G \leq \mathrm{GL}(d, \mathbb{R})$ with character γ.
- If some $P(G, v)$ full-dimensional, then $[\gamma, \chi] \leq \chi(1)$ for all $\chi \in \operatorname{Irr} G$.
- Define A, $B \subseteq \operatorname{Irr} G$ and characters α, β by

$$A := \{\chi \in \operatorname{Irr} G \mid [\gamma, \chi] = \chi(1)\}, \qquad \alpha := \sum_{\chi \in A} \chi(1)\chi,$$

$$B := \{\chi \in \operatorname{Irr} G \mid 1 \leq [\gamma, \chi] < \chi(1)\}, \qquad \beta := \gamma - \alpha.$$

  (α is the "ideal part" and β the "non-ideal part" of γ. We have $\gamma = \alpha + \beta$.)

- Set $N = \ker \beta = \bigcap_{\chi \in B} \ker \chi$.

Computing generic symmetries from the character

Theorem

Situation:

- $G \leq \mathrm{GL}(d, \mathbb{R})$ *finite matrix group*
- *with character* $\gamma = \alpha + \beta$, *"ideal part"* α.
- $N := \ker \beta$.
- $\pi \colon G \to G$ *permutation with* $\pi(1) = 1$.

Claim: π *is in the generic symmetry group of* G *if and only if*

- $\alpha(\pi(g)^{-1}\pi(h)) = \alpha(g^{-1}h)$ *for all* $g, h \in G$, *and*
- $\pi(Ng) = Ng$ *(setwise) for all* $g \in G$.

In particular: If $N = \ker \beta = \{1\}$, then $\pi = \mathrm{id}$ and G is generically closed.