X-Raying 3-Dimensional Convex Bodies with Mirror Symmetry

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Overview

The problems of illumination and covering, and their conjectures, formulated in 1960 by Boltyanski, Hadwiger, and Gohberg, Markus, have become central problems in the fields of convex, computational, and discrete geometry, and have received much attention over the last 50 years.

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However, in dimensions three and higher, only partial results have been obtained.

X-Raying is a related problem that may yield results for the problem of Illumination.

Outline

1. Covering, Illumination and X-Raying
2. X-Raying in the Plane
3. X-Raying 3-Dimensional Convex Bodies with Mirror Symmetry

Covering

Definition

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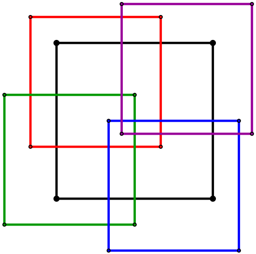

Illumination

Definition

Let $K \subset \mathbb{E}^d$ be a convex body.

- Let $p \in \mathrm{bd}(K)$ and $\mathbf{v} \in \mathbb{E}^d \setminus \{O\}$ be a direction. Then $\mathbf{v}$ is said to *illuminate* the boundary point p if the open ray emanating from p with direction $\mathbf{v}$ intersects the interior of K.

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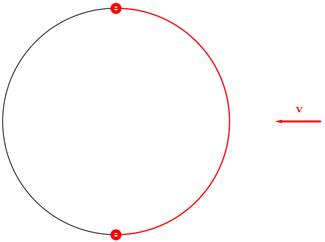
Illumination



Conjectures

Covering Conjecture (Gohberg, Markus - 1960)

Let $K \subset \mathbb{E}^d$ be a convex body. Then $C(K) \leq 2^d$, with equality if, and only if, K is an affine cube.

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Theorem

For any d-dimensional convex body K, $C(K) = I(K)$.

Two Dimensional Result

Theorem

Levi (1955) showed that for a planar convex body K,

$$C(K) = \begin{cases} 4, & \text{if } K \text{ is an affine square,} \\ 3, & \text{otherwise.} \end{cases}$$

Two Dimensional Result

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These conjectures remain open in dimensions $d > 2$. In 1999, Papadoperakis showed that $I(K) \leq 16$ for any convex body K of $\mathbb{E}^3$.

X-Raying

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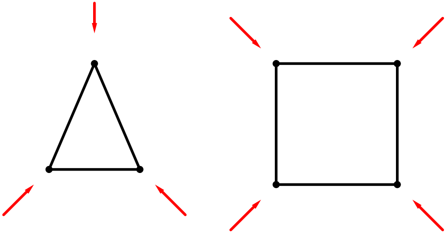

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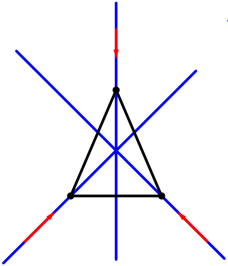

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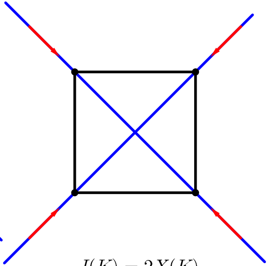

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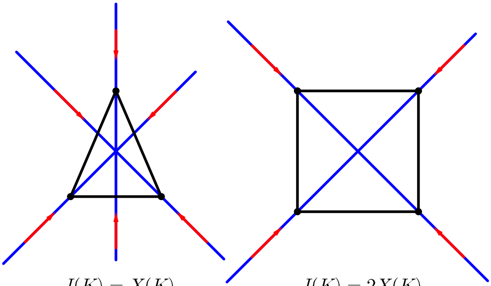

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Planar X-Ray Conjecture: First Result

From the result of Levi, we have:

Corollary

Let K be a planar convex body. Then

$$X(K) = \begin{cases} 2, & \text{if } K \text{ is an affine square,} \\ 3, & \text{otherwise.} \end{cases}$$

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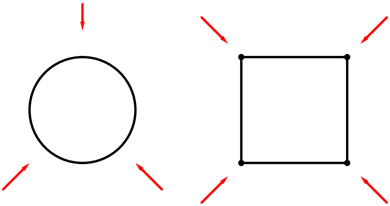

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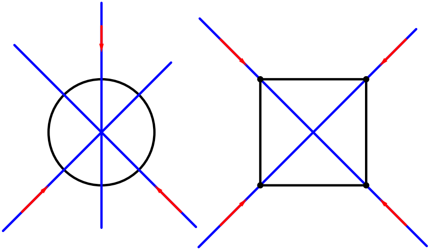

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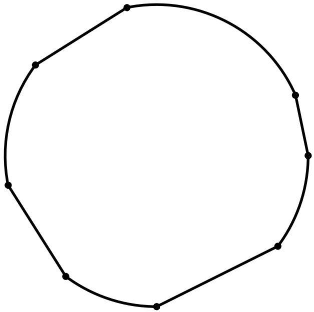

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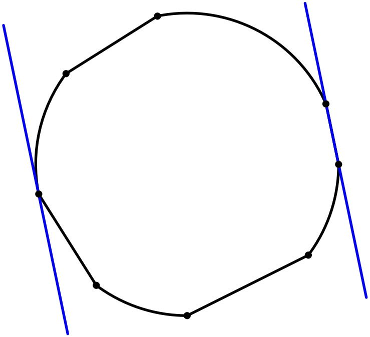

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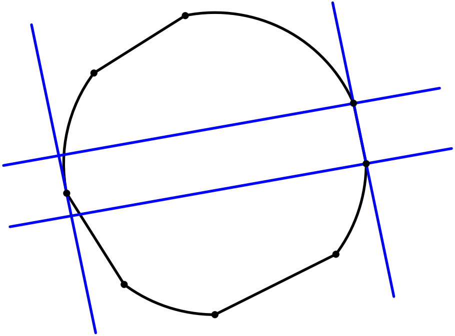

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Let K be a 3-dimensional convex body with mirror symmetry (symmetric about a plane). Without loss of generality, we may assume K is symmetric about the xy-plane.

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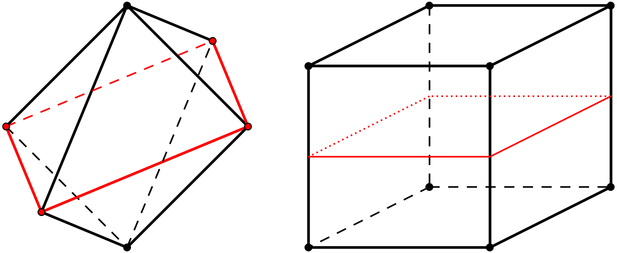
Definition

- Let B be the intersection of K with the xy-plane.
- A point $p \in \text{relbd}(B)$ is a *ground point* if $\ell^p_{\mathbf{e_3}} \cap K = \{p\}$. (here, $\mathbf{e_3} = [\begin{array}{ccc} 0 & 0 & 1 \end{array}]^T$). Otherwise, p is a *cliff point*

Ground Points and Cliff Points



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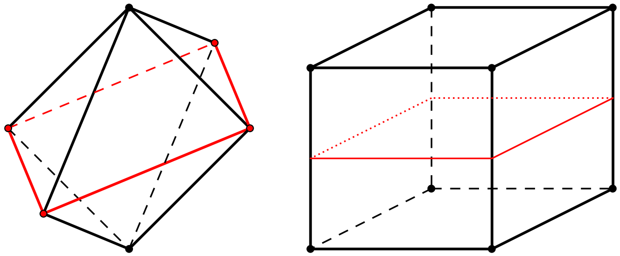
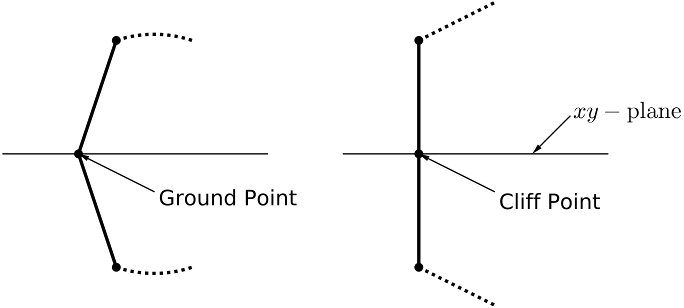


To X-Ray K, we first X-Ray B in the xy-plane, and then “tilt” the lines up and down to X-Ray K.

The Problem with Ground Points



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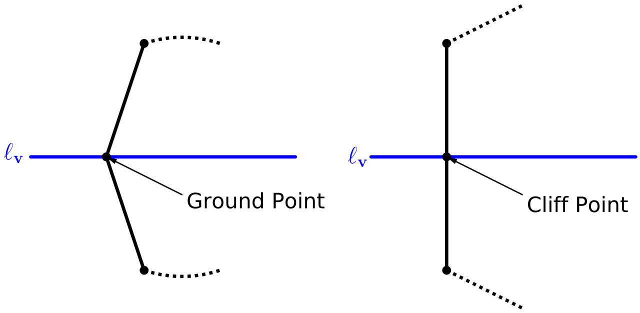

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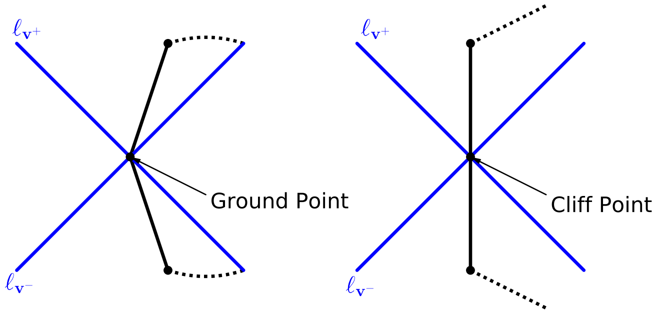

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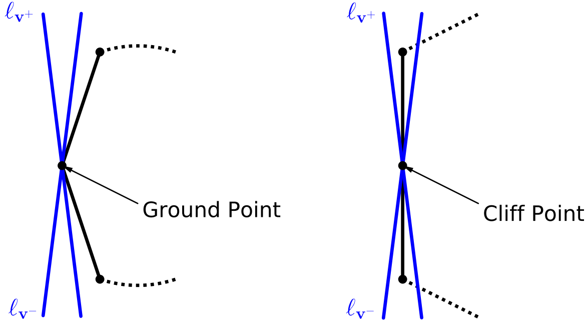

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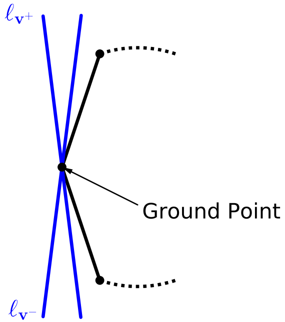

Each line $\ell_{\mathbf{v}}$ will yield two new lines, $\ell_{\mathbf{v}^+}$ and $\ell_{\mathbf{v}^-}$ but we must *keep* $\ell_{\mathbf{v}}$ to account for ground points.

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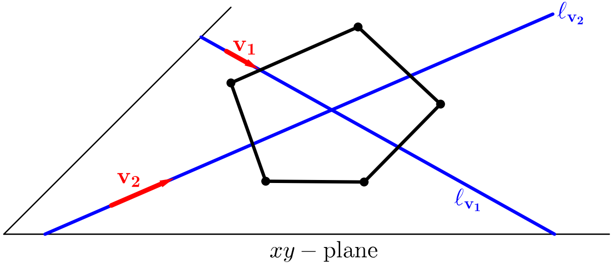

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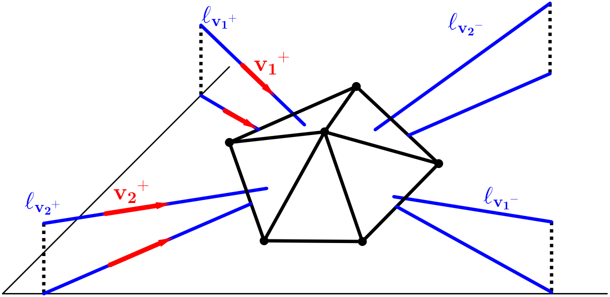

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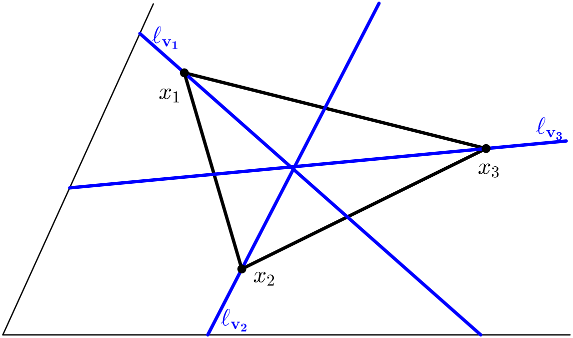

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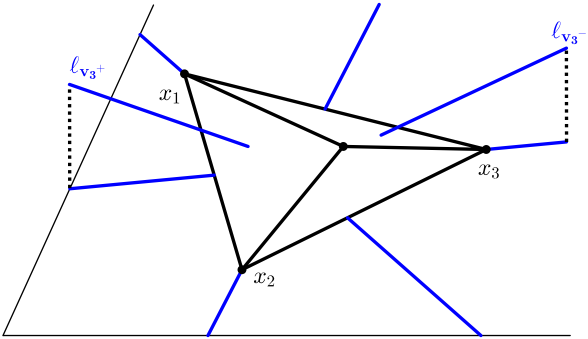

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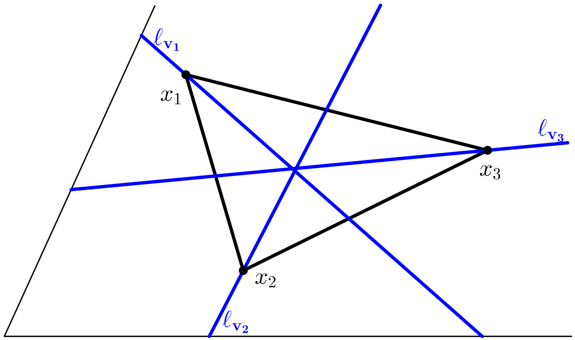

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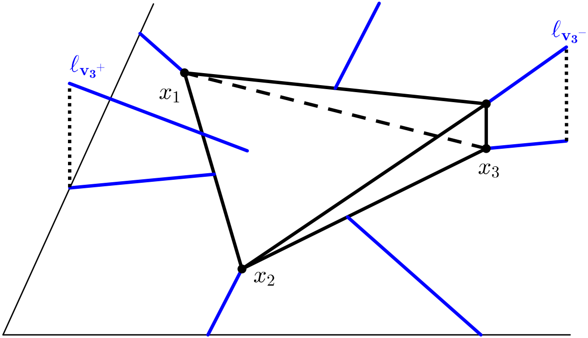

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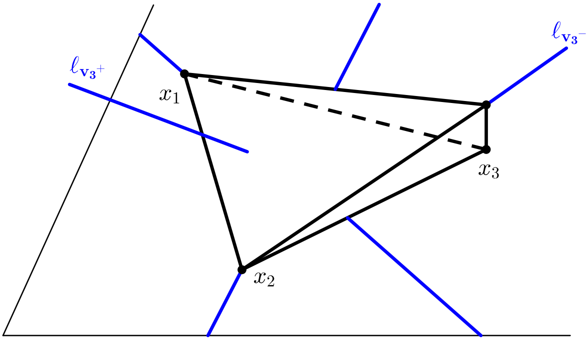
CASE II: x_1, x_2 are ground points, x_3 is a cliff point. $X(K) \leq 4$.



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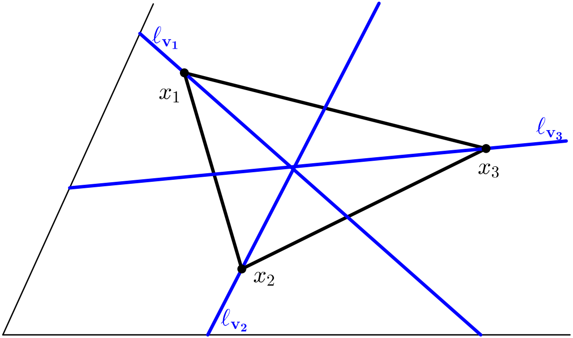

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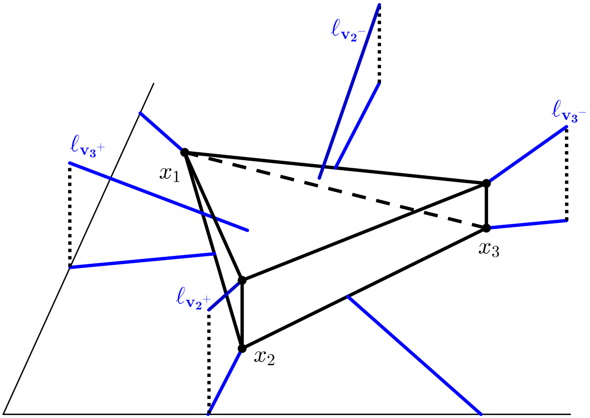

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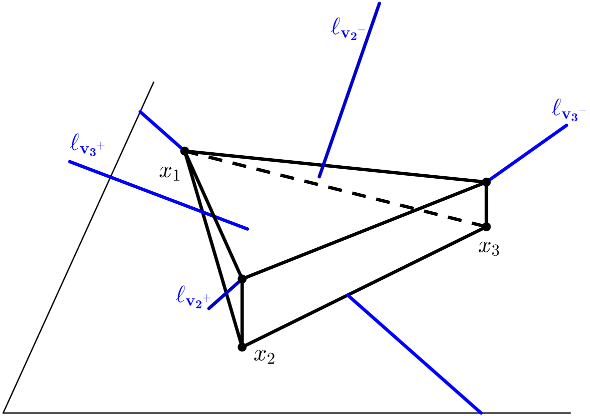

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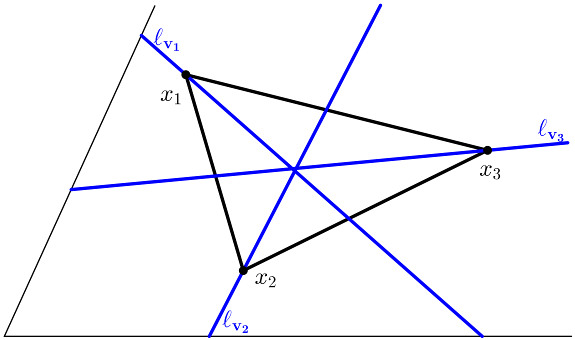

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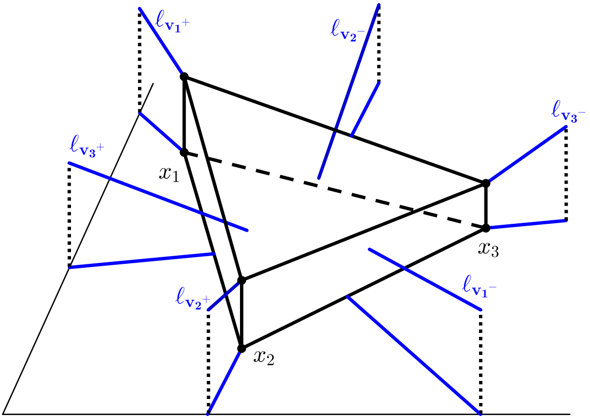

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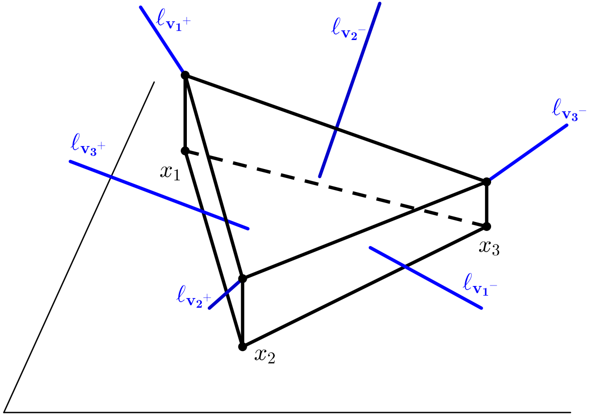

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- May lead to better bounds on the illumination number: for $d = 3$, if $X(K) \leq 6$ then $I(K) \leq 2 \cdot 6 = 12 < 16$.
- The lines that X-Ray a convex body K give rise to pairs of opposite directions that illuminate K (i.e. a starting point to find $I(K)$).

Thank You